GENERICALLY MULTIPLE TRANSITIVE ALGEBRAIC GROUP ACTIONS

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Abstract. With every nontrivial connected algebraic group G we associate a positive integer $\operatorname{gtd}(G)$ called the generic transitivity degree of G and equal to the maximal n such that there is a nontrivial action of G on an irreducible algebraic variety X for which the diagonal action of G on X^n admits an open orbit. We show that $\operatorname{gtd}(G) \leqslant 2$ (respectively, $\operatorname{gtd}(G) = 1$) for all solvable (respectively, nilpotent) G, and we calculate $\operatorname{gtd}(G)$ for all reductive G. We prove that if G is nonabelian reductive, then the above maximal n is attained for $X = G/P$ where P is a proper maximal parabolic subgroup of G (but not only for such homogeneous spaces of G). For every reductive G and its proper maximal parabolic subgroup P, we find the maximal r such that the diagonal action of G (respectively, a Levi subgroup L of P) on $(G/P)^r$ admits an open G-orbit (respectively, L-orbit). As an application, we obtain upper bounds for the multiplicities of trivial components in some tensor product decompositions. As another application, we classify all the pairs (G, P) such that the action of G on $(G/P)^3$ admits an open orbit, answering a question of M. Burger.

1. Introduction

My starting point was the following question posed to me by M. Burger in the fall of 2003.

Question 1. *Let S be a complex connected simple linear algebraic group and let P be its proper maximal parabolic subgroup. For which S and P is there an open S-orbit in $S/P \times S/P \times S/P$?*

Answering this question led me to the following general set up. Consider an algebraic action of an algebraic group G on an algebraic variety X. Its n-transitivity means that

(i) for the diagonal action of G on $X^n := \underbrace{X \times \ldots \times X}_{n}$, there is an open G-orbit $\mathcal{O}$;

(ii) $X^n \setminus \mathcal{O}$, the boundary of $\mathcal{O}$, is the union of "diagonals" of X^n.

All multiple transitive (i.e., with $n \geqslant 2$) actions are classified, [Kn]. They constitute a rather small and not very impressive class: for $n \geqslant 4$, there are none of them; for $n = 3$, it is only the natural action of $\mathbf{PGL}_2$ on $\mathbf{P}^1$; for $n = 2$ and reductive G, it is only the natural action of $\mathbf{PGL}_{m+1}$ on $\mathbf{P}^m$.

From the point of view of algebraic transformation groups, imposing restriction (ii) does not look really natural. It is more natural to consider the cases where G acts transitively on all n-tuples of elements of X subject to algebraic inequalities *depending* on the problem under consideration. This leads to the following definition.

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Definition 1. Let n be a positive integer. An algebraic action $\alpha : G \times X \to X$ of a connected algebraic group G on an irreducible algebraic variety X is called *generically n-transitive* if the diagonal action of G on X^n is locally transitive, i.e., admits an open G-orbit. If α is not locally transitive, then α is called *generically 0-transitive.*

Below we will see that the class of generically multiple transitive actions is much more rich and interesting than that of multiple transitive ones.

Since the projection $X^n \to X^{n-1}$, $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$ is G-equivariant, generic n-transitivity yields generic m-transitivity for $0 < m \leqslant n$. As $\dim X^n = n \dim X$,

$$\alpha \text{ is not generically } n\text{-transitive if } n \dim X > \dim G. \tag{1}$$

Definition 2. The *generic transitivity degree of an action α* is

$$\mathrm{gtd}(\alpha) := \sup n$$

with the supremum taken over all n such that α is generically n-transitive. The *generic transitivity degree of a nontrivial connected algebraic group G* is

$$\mathrm{gtd}(G) := \sup \mathrm{gtd}\,(\alpha)$$

with the supremum taken over all nontrivial actions α of G on irreducible algebraic varieties.

Thus $\mathrm{gtd}(\alpha)$ is a positive integer or $+\infty$ and the latter holds if and only if X is a single point. Since every nontrivial group G admits a nontrivial transitive action, $\mathrm{gtd}(G)$ is a well defined positive integer, and by (1), we have $\mathrm{gtd}(G) \leqslant \dim G$.

Using this terminology, Question 1 can be reformulated as the problem of classifying all proper maximal parabolic subgroups P of connected simple algebraic groups G such that the action of G on G/P is generically 3-transitive.

In this paper I address the problem of calculating the generic transitivity degrees of connected linear algebraic groups. The main results are the following Theorems 1–6.

Theorem 1. *Let G be a nontrivial connected linear algebraic group.*

(i) *If G is solvable, then*
$$\mathrm{gtd}(G) \leqslant 2.$$

(ii) *If G is nilpotent, then*
$$\mathrm{gtd}(G) = 1.$$

(iii) *If G is reductive and $\widetilde{G} \to G$ is an isogeny, then*
$$\mathrm{gtd}(\widetilde{G}) = \mathrm{gtd}(G).$$

(iv) *If Z is a torus and S_i a connected simple algebraic group, $i = 1, \ldots, d$, then*
$$\mathrm{gtd}(Z \times S_1 \times \ldots \times S_d) = \max_i \mathrm{gtd}(S_i).$$

(v) *If G is simple, then* $\mathrm{gtd}(G)$ *is given by Table* 1*:*

type of G	A_l	B_l, $l \geqslant 3$	C_l, $l \geqslant 2$	D_l, $l \geqslant 4$	E_6	E_7	E_8	F_4	G_2
$\mathrm{gtd}(G)$	$l+2$	3	3	3	4	3	2	2	2

Table 1

Theorem 2. *Let G be a connected nonabelian reductive group. Then there is a proper maximal parabolic subgroup P of G such that the generic transitivity degree of the natural action of G on G/P is equal to* $\mathrm{gtd}(G)$.

Given two positive integers i and l, $i \leqslant l$, we put

$$\begin{aligned} \mathcal{M}_{li} &:= \Big\{a \in \mathbb{N} \mid a < \frac{(l+1)^2}{i(l+1-i)}\Big\}, \\ m_{li} &:= \max_{a\in\mathcal{M}_{li}} a. \end{aligned} \tag{2}$$

Theorem 3. *If G is a simple linear algebraic group, then the generic transitivity degree* $\mathrm{gtd}(G : G/P_i)$ *of the natural action of G on G/P_i, where P_i is a standard proper maximal parabolic subgroup of G corresponding to the i-th simple root* (*see Section* 4 *below*), *is given by Table* 2*:*

type of G	$\mathrm{gtd}(G : G/P_i)$
A_l	m_{li} (see (2))
B_l, $l \geqslant 3$	2 if $i \neq 1, l$, 3 if $i = 1, l$
C_l, $l \geqslant 2$	2 if $i \neq 1, l$, 3 if $i = 1, l$
D_l, $l \geqslant 4$	2 if $i \neq 1, l-1, l$, 3 if $i = 1, l-1, l$
E_6	4 if $i = 1, 6$, 2 if $i \neq 1, 6$
E_7	2 if $i \neq 7$, 3 if $i = 7$
E_8	2
F_4	2
G_2	2

Table 2

Since (2) yields $m_{li} \geqslant 3$ for all i, l, and $m_{li} = 3$ if and only if $2i = l + 1$, we obtain

Corollary 1. *Let G be a connected simple algebraic group of type A_l. Then*

$$\mathrm{gtd}(G : G/P_i) \begin{cases} = 3 & \text{if } 2i = l+1, \\ \geqslant 4 & \text{otherwise.} \end{cases}$$

The next corollary answers M. Burger's Question 1.

Corollary 2. *Maintain the notation of Question* 1*. The following properties are equivalent*:

(i) $(S/P)^3$ *contains an open S-orbit*;

(ii) *S is of type A_l, B_l, C_l, D_l, E_6 or E_7, and P is conjugate to one of the standard proper maximal parabolic subgroups P_i of S given by Table* 3*:*

type of G	A_l	B_l, $l \geqslant 3$	C_l, $l \geqslant 2$	D_l, $l \geqslant 4$	E_6	E_7
i	$1, \ldots, l$	$1, l$	$1, l$	$1, l-1, l$	$1, 6$	7

Table 3

Remark 1. Theorem 3 shows that, in the notation of Question 1, there are the cases where S has an open orbit in $(S/P)^d$ not only for $d = 3$ but also for $d = 4$ (for S of types A_l with $l \geqslant 2$, and E_6) and $d \geqslant 5$ (for S of type A_l with $l \geqslant 3$).

Remark 2. It would be interesting to extend Theorem 3 by calculating for every simple G the generic transitivity degree $\mathrm{gtd}(G : G/P)$ of the action of G on G/P for every non-maximal parabolic subgroup P. By Lemma 2 below, if B is a Borel subgroup of G, then $\mathrm{gtd}(G : G/P) \geqslant \mathrm{gtd}(G : G/B)$ for every P. By Corollary 2 of Proposition 2 below, $\mathrm{gtd}(G : G/B) = 2$ (respectively, 3) if G is not (respectively, is) of type A_1. This and Theorem 3 yield that $\mathrm{gtd}(G : G/P) = 2$ for every P if G is of types E_8, F_4 or G_2. $\square$

According to classical Richardson's theorem, [Ri$_1$], if $P = LU$ is a proper parabolic subgroup of a connected reductive group G with U the unipotent radical of P and L a Levi subgroup in P, then for the conjugating action there is an open P-orbit in U. In general, an open L-orbit in U may not exist. A standard argument (see Section 4) reduces the problem of classifying cases where there is an open L-orbit in U to that for simple G. If P is maximal, such a classification is obtained as a byproduct of our proof of the above theorems:

Theorem 4. *Maintain the above notation. Let P^- be a parabolic subgroup opposite to P.*

(a) *The following properties are equivalent:*
  (i) *U contains an open L-orbit,*
  (ii) *G/P contains an open L-orbit,*
  (iii) *$(G/P)^2 \times G/P^-$ contains an open G-orbit.*

(b) *If G is simple and P is maximal, then properties* (i), (ii), (iii) *hold if and only if P is conjugate to one of the standard maximal parabolic subgroups P_i of G given by Table* 4*:*

type of G	A_l	B_l, $l \geqslant 3$	C_l, $l \geqslant 2$	D_l, $l \geqslant 4$	E_6	E_7
i	$1, \ldots, l$	$1, l$	$1, l$	$1, l-1, l$	$1, 3, 5, 6$	7

Table 4

Remark 3. Finding reductive subgroups of G that act locally transitively on G/P is given much attention in the literature, see a survey in [Kime$_2$] and the references therein. Note that (ii) in Theorem 4 is equivalent to a certain representation theoretic property of the triple (G, L, P) (simplicity of the spectrum), [VK]. $\square$

Actually, for maximal P, we calculate the generic transitivity degrees $\mathrm{gtd}(L : G/P)$ and $\mathrm{gtd}(L : U)$ of the L-actions on G/P and U. In order to formulate the answer, we put for every two positive integers i and l, $i \leqslant l$,

$$\mathcal{S}_{li} := \Big\{ a \in \{2, 3, \ldots\} \mid \tfrac{i}{l+1-i} \notin \Big[\tfrac{a-\sqrt{a^2-4}}{2} \ , \ \tfrac{a+\sqrt{a^2-4}}{2} \Big] \Big\}. \qquad (3)$$

We then have $\mathcal{S}_{li} = \varnothing$ if and only if $2i = l+1$. For $2i \neq l+1$, we put

$$s_{li} := \max_{a \in \mathcal{S}_{li}} a. \tag{4}$$

Theorem 5. *Maintain the above notation.*

(i) $\mathrm{gtd}(L : G/P) = \mathrm{gtd}(L : U)$.

(ii) *If G is simple, then $\mathrm{gtd}(L : G/P_i)$ is given by Table 5:*

type of G	$\mathrm{gtd}(L : G/P_i)$
A_l	1 if $2i = l+1$, s_{li} if $2i \neq l+1$ (see (4),(3))
B_l, $l \geqslant 3$	0 if $i \neq 1, l$, 1 if $i = 1, l$
C_l, $l \geqslant 2$	0 if $i \neq 1, l$, 1 if $i = 1, l$
D_l, $l \geqslant 4$	0 if $i \neq 1, l-1, l$, 1 if $i = 1$, 1 if l is even and $i = l-1, l$, 2 if l is odd and $i = l-1, l$
E_6	0 if $i = 2, 4$, 1 if $i = 3, 5$, 2 if $i = 1, 6$
E_7	0 if $i \neq 7$, 1 if $i = 7$
E_8	0
F_4	0
G_2	0

Table 5

Finally, we show that calculating the generic transitivity degrees of actions on generalized flag varieties is closely related to the problem of decomposing tensor products.

Namely, let G be a connected simply connected semisimple algebraic group. Fix a Borel subgroup B of G and a maximal torus T of B. Let P_{++} be the additive monoid of dominant weights of T determined by B. For $\varpi \in \mathrm{P}_{++}$, denote by $E(\varpi)$ a simple G-module of highest weight ϖ, and by ϖ^* the highest weight of the dual G-module $E(\varpi)^*$. Let $P(\varpi)$ be the G-stabilizer of the unique B-stable line in $E(\varpi)$. The subgroup $P(\varpi)$ of G is parabolic; every parabolic subgroup of G is obtained this way. If ϖ is fundamental, then $P(\varpi)$ is maximal.

Theorem 6. *Maintain the above notation. Let d be a positive integer and let $\varpi \in \mathrm{P}_{++}$.*

(i) *if* $\mathrm{gtd}(G : G/P(\varpi)) \geqslant d$, *then*

$$\dim\big(E(n_1\varpi^*) \otimes \ldots \otimes (E(n_d\varpi^*)\big)^G \leqslant 1 \quad \textit{for every } n_1, \ldots, n_d \in \mathbb{Z}_+; \tag{5}$$

(ii) *if ϖ is fundamental, the converse it true, i.e.,* (5) *yields* $\operatorname{gtd}(G : G/P(\varpi)) \geqslant d$.

As an application of Theorems 6, 3, we obtain upper bounds of the multiplicities of trivial components in some tensor product decompositions.

Example 1. Let ϖ_i be the ith fundamental weight of $G = \mathbf{SL}_m$. Then

$$\dim\bigl(E(n_1\varpi_i) \otimes \ldots \otimes (E(n_d\varpi_i)\bigr)^G \leqslant 1 \quad \text{for every } n_1, \ldots, n_d \in \mathbb{Z}_+$$

if and only if $d < m^2/(im - i^2)$.

In [P] we develop further the latter topic.

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2. Conventions, notation, and terminology

In this paper the characteristic of the base algebraically closed field k is equal to 0. The reason is that in some of the proofs I use the classification results from [Kimu], [KKIY], [KKTI] that are obtained under this constraint on $\operatorname{char} k$. The problem of extending the main results to positive characteristic (perhaps with some small primes excluded) looks manageable. I did not attempt to solve it here.

Below every action of algebraic group is algebraic (morphic).

Given actions of G on $X_1, \ldots, X_n$, the action of G on $X_1 \times \ldots \times X_n$ means the diagonal action.

Given the subgroups S and H of G, the action of S on G/H is always the natural action induced by left translations. It is denoted by $(S : G/H)$.

Let $P = LU$ be a parabolic subgroup of G with U the unipotent radical and L a Levi subgroup in P, and let $\mathfrak{u}$ be the Lie algebra of U. Then $(L : U)$ and $(L : \mathfrak{u})$ denote the conjugating and adjoint actions of L on U and $\mathfrak{u}$ respectively.

Given an action of a group G on a set X, we denote by $G \cdot x$ and G_x respectively the G-orbit and G-stabilizer of a point $x \in X$. The fixed point set G on X is denoted by X^G.

Throughout the paper the notion of *general point* is used. By that it is meant a point lying off a suitable closed subvariety.

G^0 is the identity component of an algebraic group G.

(G, G) is the commutator group of a group G.

Given a root system with a base $\Delta = \{\alpha_1, \ldots, \alpha_r\}$, we enumerate the simple roots $\alpha_1, \ldots, \alpha_r$ as in [Bou]. If $\varpi_1, \ldots, \varpi_r$ is the system of fundamental weights corresponding to Δ and $\lambda = a_1\varpi_1 + \ldots + a_r\varpi_r$ is a weight, then we write $\lambda = (a_1, \ldots, a_r)$. The labelled Dynkin diagram of Δ, where the label of i-th vertex is a_i, will be called the *Dynkin diagram of* λ (if $a_i = 0$, then the i-th label is dropped). Note that if $\lambda = c_1\alpha_1 + \ldots + c_r\alpha_r$, then $(a_1, \ldots, a_r)$ is the linear combination with coefficients $c_1, \ldots, c_r$ of the rows of the Cartan matrix of Δ.

We call a connected linear algebraic group *simple* if it has no proper closed normal subgroups of positive dimension.

k^n is the n-dimensional coordinate space of column vectors over k.

$k[X]$ is the algebra of regular functions of an algebraic variety X.

$k(X)$ is the field of rational functions of an irreducible algebraic variety X.

We put $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\mathbb{N} = \{1, 2, \ldots\}$, and $[a, b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$.

3. Properties of generically multiple transitive actions

In this section we establish some general properties of generically multiple transitive actions that will be used in the proof of Theorems 1–4.

Let $\sigma : G \to H$ be a surjective homomorphism of nontrivial connected algebraic groups. Let α be an action of H on an irreducible variety X, and let ${}^{\sigma}\alpha$ be the action of G on X defined by

$$ {}^{\sigma}\alpha(g, x) := \alpha(\sigma(g), x). \tag{6} $$

Lemma 1. *Maintain the above notation. Then*

(i) $\operatorname{gtd}({}^{\sigma}\alpha) = \operatorname{gtd}(\alpha)$,

(ii) $\operatorname{gtd}(G) \geqslant \operatorname{gtd}(H)$.

Proof. (i) is clear, and (ii) follows from (i) and Definition 2. □

Lemma 2. *Let α_i be an action of a connected algebraic group G on an irreducible algebraic variety X_i, $i = 1, 2$. Assume that there exists a G-equivariant dominant rational map $\varphi : X_1 \dashrightarrow X_2$.*

(i) $\operatorname{gtd}(\alpha_1) \leqslant \operatorname{gtd}(\alpha_2)$.

(ii) *If* $\dim X_1 = \dim X_2$, *then* $\operatorname{gtd}(\alpha_1) = \operatorname{gtd}(\alpha_2)$.

Proof. (i) Assume that the action of G on X_1^n is locally transitive. Since the rational map $\varphi^n : X_1^n \dashrightarrow X_2^n$, $(x_1, \ldots, x_n) \mapsto (\varphi(x_1), \ldots, \varphi(x_n))$, is G-equivariant and dominant, the indeterminacy locus of φ^n lies in the complement to the open G-orbit in X_1^n, and the image of this orbit under φ^n is a G-orbit open in X_2^n. Definitions 1, 2 now yield that (i) holds.

(ii) Assume that $\dim X_1 = \dim X_2$ and there is an open G-orbit $\mathcal{O}$ in X_2^n. Since φ^n is G-equivariant and dominant, there is a point $z \in X_1^n$ such that the orbit $G \cdot z$ lies off the indeterminacy locus of φ^n and $\varphi^n(G \cdot z) = \mathcal{O}$. This yields $\dim X_2^n = \dim X_1^n \geqslant \dim G \cdot z \geqslant \dim \mathcal{O} = \dim X_2^n$. Hence $\dim X_1^n = \dim G \cdot z$, i.e., $G \cdot z$ is open in X_1^n. From (i) and Definitions 1, 2 we now deduce that (ii) holds. □

Corollary. *The generic transitivity degree is a birational invariant of actions, i.e., if φ in Lemma 2 is a birational isomorphism, then* $\operatorname{gtd}(\alpha_1) = \operatorname{gtd}(\alpha_2)$. □

Lemma 3. *Let a connected algebraic group G_i act on an irreducible variety X_i, $i = 1, \ldots, d$. All these actions are generically n-transitive if and only if the natural action of $G_1 \times \ldots \times G_d$ on $X_1 \times \ldots \times X_d$ is generically n-transitive.*

Proof. This easily follows from Definition 1. □

Lemma 4. *Let an algebraic group G act on irreducible algebraic varieties X and Y.*

(a) *The following properties are equivalent:*

  (i) *The action of G on $X \times Y$ is locally transitive.*

  (ii) *The action of G on X is locally transitive and if H is the G-stabilizer of a general point of X, then the natural action of H on Y is locally transitive.*

(b) *Assume that* (i) *and* (ii) *hold. If $x \in X$ and $y \in Y$ are the points such that the orbits $G \cdot x$ and $G_x \cdot y$ are open in X and Y respectively, then the orbit $G \cdot (x, y)$ is open in $X \times Y$.*

Proof. Note that for every point $z = (x, y) \in X \times Y$, we have

$$G_z = (G_x)_z = (G_x)_y. \tag{7}$$

Assume that (i) holds and the orbit $G \cdot z$ is open in $X \times Y$. As the natural projection $\pi_X : X \times Y \to X$ is G-equivariant and surjective, the orbit $G \cdot x = \pi_X(G \cdot z)$ is open in X. The fiber $\pi_X^{-1}(x)$ is G_x-stable, contains z, and $G_x \cdot z = G \cdot z \cap \pi_X^{-1}(x)$. Hence the orbit $G_x \cdot z$ is open in $\pi_X^{-1}(x)$. As the natural projection $\pi_X^{-1}(x) \to Y$ is a G_x-equivariant isomorphism, (ii) and (b) follow from (7).

Conversely, assume that (ii) holds. Then, if the above point z is general, (7) yields $\dim G\cdot z = \dim G - \dim G_z = \dim G - \dim (G_x)_y = (\dim G - \dim G_x) + (\dim G_x - \dim (G_x)_y) = \dim X + \dim Y = \dim(X \times Y)$, i.e., $G \cdot z$ is open in $X \times Y$. So (i) holds. $\square$

Lemma 5. *Let P be a proper parabolic subgroup of a connected reductive group G. Let w_0 be the element of the Weyl group of G with maximal length (with respect to a system of simple reflections), and let $\dot{w}_0$ be a representative of w_0 in the normalizer of a maximal torus of P. Let $x \in G/P$ be a point corresponding to the coset P. Then the G-orbit of point $z := (x, \dot{w}_0 \cdot x)$ is open in $(G/P)^2$.*

Proof. It follows from Bruhat decomposition, cf. [Bor, 14.12, 14.14], that $P\dot{w}_0P$ is open in G. Hence the P-orbit of $\dot{w}_0 \cdot x$ is open in G/P. Since $G_x = P$, the claims now follows from Lemma 4 and (7). $\square$

Corollary. *Let α be the action of a connected linear algebraic group G on G/P, where P is a proper parabolic subgroup of G. Then* $\mathrm{gtd}(\alpha) \geqslant 2$.

Proof. Since P contains the radical of G, we may assume that G is reductive and then apply Lemma 5. $\square$

Proposition 1. *Let G be a nontrivial connected linear algebraic group.*

(i) *If G is nonsolvable, then* $\mathrm{gtd}(G) \geqslant 2$.
(ii) *If G is solvable, then* $\mathrm{gtd}(G) \leqslant 2$.
(iii) *If G is nilpotent, then* $\mathrm{gtd}(G) = 1$.

Proof. Since G is nonsolvable if and only if it contains a proper parabolic subgroup, [Sp, 6.2.5], statement (i) follows from the Corollary of Lemma 5 and Definition 2.

Assume now that G is solvable (respectively, nilpotent). Let α be an action of G on an irreducible algebraic variety X such that

$$\mathrm{gtd}(G) = \mathrm{gtd}(\alpha). \tag{8}$$

Since $\mathrm{gtd}(G) \geqslant 1$, there is an open G-orbit in X. By the Corollary of Lemma 2, we may, maintaining equality (8), replace X by this orbit. So we may (and shall) assume that α is the action of G on $X = G/H$ where H is a proper closed subgroup of G. If Q is a proper maximal closed subgroup of G containing H, then the existence of the natural G-equivariant morphism $G/H \to G/Q$, Lemma 2(i), Definition 2, and equality (8) yield that we may, maintaining (8), replace H by Q. So we may (and shall) assume that H is a proper maximal closed subgroup of G. If H contains a nontrivial closed normal subgroup N of G, then N acts trivially on X, so maintaining the generic transitivity degree of the action on X and the assumption that G is solvable (respectively, nilpotent), the group G may be replaced by G/N. So we may (and shall) also assume that H contains no nontrivial closed normal subgroups of G. Finally, by Lemma 2(ii), replacing H by H^0, we may (and shall) assume that H is connected.

Use now that in every nontrivial connected nilpotent linear algebraic group the dimension of center is positive, and the dimension of normalizer of every proper subgroup is strictly bigger than that of this subgroup, cf. [Hu, 17.4]. From this and the properties of H we deduce that if G is nilpotent, then $\dim G = 1$ and H is trivial. By (1), we then obtain $\mathrm{gtd}(G) = 1$. This proves (iii).

Assume now that G is not nilpotent (but solvable). Then the set G_u of all unipotent elements of G is a nontrivial closed connected normal subgroup of G, and if T is a maximal torus of G, then G is a semidirect but not direct product of T and G_u, see [Sp, 6.3]. Since H is a connected solvable group as well, if S is maximal torus of H, then H is a semidirect product of S and H_u. We have $H_u \subseteq G_u$ and we may (and shall) take S and T so that $S \subseteq T$. Since H contains no nontrivial closed normal subgroups of G, we have $H_u \neq G_u$. Therefore if $S \neq T$, then SG_u is a proper closed subgroup of G containing H. As H is maximal, this is impossible. Thus $S = T$.

Consider now the center Z_{G_u} of G_u. By [Sp, 6.3.4], we have $\dim Z_{G_u} \geqslant 1$. Since Z_{G_u} is a closed normal subgroup of G, it is stable with respect to the conjugating action of T. Since Z_{G_u} is a commutative unipotent group, it is T-equivariantly isomorphic to the vector group of its Lie algebra $\mathfrak{z}_{G_u}$ on which T acts via the adjoint representation (if we embed G in some $\mathbf{GL}_n$, the exponential map is a T-equivariant isomorphism of the last group to Z_{G_u}). As the T-module $\mathfrak{z}_{G_u}$ is a direct sum of one-dimensional submodules, from this we deduce that Z_{G_u} contains a one-dimensional closed subgroup U normalized by T.

Since $H = TH_u$, the subgroup U is normalized by H as well. Note now that the subgroup $H_u \cap Z_{G_u}$ is trivial. Indeed, since T normalizes H_u and $G = TG_u$, this subgroup is normal in G, and as H contains no nontrivial closed normal subgroups of G, the claim follows. From this we deduce that the subgroup $H_u \cap U$ is trivial. Since $\dim U = 1$ and U is normalized by H, this easily yields that HU is a closed subgroup of dimension $\dim H + 1$. The maximality of H then yields $HU = G$. We now conclude that the variety $X = G/H$ is isomorphic to the affine line $\mathbf{A}^1$.

It is well known that $\mathrm{Aut}\,\mathbf{A}^1$ coincides with the group $\mathbf{Aff}_1$ of all affine transformations of $\mathbf{A}^1$. Hence the action of G on X induces a homomorphism $\varphi : G \to \mathbf{Aff}_1$. We have $\ker\varphi \subseteq H$. Since H contains no nontrivial closed normal subgroups of G, this yields that φ is injective. As $\mathbf{Aff}_1$ is a connected 2-dimensional linear algebraic group, every its proper subgroup is abelian, hence nilpotent. Since G is not nilpotent, from this we deduce that φ is an isomorphism. It now remains to note that using Lemma 4(a) and (1) one easily verifies that the the generic transitivity degree of the action of $\mathbf{Aff}_1$ on $\mathbf{A}^1$ is equal to 2. This proves (ii). $\square$

Let $P = LU$ be a proper parabolic subgroup of a connected reductive group G with U the unipotent radical of P and L a Levi subgroup in P. Let P^- be the unique (cf. [Bor, 14.21]) parabolic subgroup opposite to P and containing L, and let U^- be the unipotent radical of P^-. Denote by $p \in G/P$ and $p^- \in G/P^-$ the points corresponding to the cosets P and P^-. The orbits $U^- \cdot p$ and $U \cdot p^-$ are L-stable and open in G/P and G/P^-, [Bor, 14.21]. Let $\mathfrak{u}$ and $\mathfrak{u}^-$ be the Lie algebras of U and U^-.

Proposition 2. *Maintain the above notation.*

(i) *U, $\mathfrak{u}$, $U \cdot p^-$ are L-isomorphic varieties; the same holds for U^-, $\mathfrak{u}^-$, $U^- \cdot p$. So,* $\mathrm{gtd}(L : U) = \mathrm{gtd}(L : \mathfrak{u})$, $\mathrm{gtd}(L : U^-) = \mathrm{gtd}(L : \mathfrak{u}^-)$.

(ii) $(L : \mathfrak{u}^-) = {}^{\sigma}(L : \mathfrak{u})$ *(see* (6)*) for some $\sigma \in \mathrm{Aut}\,L$, and* $\mathrm{gtd}(L : \mathfrak{u}) = \mathrm{gtd}(L : \mathfrak{u}^-)$.

(iii) $\mathrm{gtd}(L : U) = \mathrm{gtd}(L : G/P) = \mathrm{gtd}(L : U^-) = \mathrm{gtd}(L : G/P^-)$.

(iv) $(G/P)^d \times G/P^-$ *contains an open* G*-orbit if and only if* $d \leqslant 1 + \operatorname{gtd}(L : U^-)$.

Proof. (i) We may assume that $G \subseteq \mathbf{GL}_n$, [Sp, 2.3.7]. Then elements of U and $\mathfrak{u}$ are respectively unipotent and nilpotent matrices. Hence the L-equivariant morphisms $\mathfrak{u} \to U$, $Y \mapsto \sum_{i=0}^{n-1} \frac{1}{i!} Y^i$, and $U \to \mathfrak{u}$, $Z \mapsto -\sum_{i=1}^{n-1} \frac{1}{i}(I_n - Z)^i$ are inverse to one another. Thus U and $\mathfrak{u}$ are L-isomorphic. Since $U \to U \cdot p^-$, $u \mapsto u \cdot p^-$, is an L-equivariant isomorphism, cf. [Bor, 14.21], U is L-isomorphic to $U \cdot p^-$. For U^-, $\mathfrak{u}^-$, $U^- \cdot p$ the arguments are the same.

(ii) The first claim follows from the fact that L is a connected reductive group and the L-modules $\mathfrak{u}$ and $\mathfrak{u}^-$ are dual to one another (cf., e.g., [Röh] and Section 4). The second follows from the first and Lemma 1.

(iii) Since $U^- \cdot p$ is open in G/P, we deduce from the Corollary of Lemma 2 and (i) that $\operatorname{gtd}(L : G/P) = \operatorname{gtd}(L : U^-)$. Analogously, $\operatorname{gtd}(L : G/P^-) = \operatorname{gtd}(L : U)$. The claim now follows from (i) and (ii).

(iv) Since $G_{p^-} = P^-$ and $P^- \cdot p = U^- L \cdot p = U^- \cdot p$, the orbit $G_{p^-} \cdot p$ is open in G/P. Lemma 4 then yields that the orbit $G \cdot z$, where $z = (p, p^-) \in G/P \times G/P^-$, is open in $G/P \times G/P^-$. Since $G_z = P \cap P^- = L$, applying Lemma 4 again, we deduce that $(G/P)^d \times G/P^-$ for $d \geqslant 2$ contains an open G-orbit if and only if $(G/P)^{d-1}$ contains an open L-orbit. By Definition 2, the latter holds if and only if $d - 1 \leqslant \operatorname{gtd}(L : G/P)$. On the other hand, since $U^- \cdot p$ is open in G/P, the Corollary of Lemma 2 and (i) yield that $\operatorname{gtd}(L : G/P) = \operatorname{gtd}(L : U^-)$. $\square$

Corollary 1. (i) $\operatorname{gtd}(G : G/P) \geqslant 1 + \operatorname{gtd}(L : \mathfrak{u}^-)$.
(ii) *If* P *and* P^- *are conjugate, then* $\operatorname{gtd}(G : G/P) = 2 + \operatorname{gtd}(L : \mathfrak{u}^-)$.

Proof. This follow from Definition 2, Proposition 2, and two remarks: (a) As the projection $(G/P)^d \times G/P^- \to (G/P)^d$ is G-equivariant, the existence of an open G-orbit in $(G/P)^d \times G/P^-$ yields its existence in $(G/P)^d$. (b) The assumption in (ii) yields $(G/P)^{d+1} = (G/P)^d \times G/P^-$. $\square$

Corollary 2. *Let* P *be conjugate to* P^-*. The following properties are equivalent:*

(i) $(G : G/P)$ *is generically* 3*-transitive,*
(ii) $(L : \mathfrak{u}^-)$ *is locally transitive.* $\square$

Remark 4. If U is abelian, then, by [Ri$_2$], the number of L-orbits in U is finite, hence, by Proposition 2, $(L : \mathfrak{u}^-)$ is locally transitive (in this case U^- is abelian as well, see Section 4, and, by Proposition 2, the number of L-orbits in $\mathfrak{u}^-$ is equal to that in $\mathfrak{u}$). The parabolic subgroups P whose unipotent radical is abelian are easy to classify, [RRS]. Every such P is maximal. If G is simple, then up to conjugacy all such P are exhausted by the following standard parabolic subgroups P_i:

type of G	A_l	B_l	C_l	D_l	E_6	E_7
i	$1, \ldots, l$	1	l	$1, l-1, l$	$1, 6$	7

Corollary 3. *Let* G *be a connected semisimple group and let* B *be a Borel subgroup of* G*.*

(a) $2 \leqslant \operatorname{gtd}(G : G/B) \leqslant 3$.
(b) *The following properties are equivalent:*
   (i) $\operatorname{gtd}(G : G/B) = 3$,
   (ii) G *is locally isomorphic to* $\mathbf{SL}_2 \times \ldots \times \mathbf{SL}_2$.

Proof. Use the above notation for $P = B$. By Lemma 5, we have $\mathrm{gtd}(G : G/B) \geqslant 2$.

Assume that $(G : G/B)$ is generically 3-transitive. Since P is conjugate to P^-, and L is a maximal torus of G, Corollary 2 yields $\dim L \geqslant \dim U^-$ = number of negative roots $\geqslant$ number of simple roots $= \dim L$. Therefore every positive root is simple, whence G is locally isomorphic to $\mathbf{SL}_2 \times \ldots \times \mathbf{SL}_2$. Now Proposition 4 below and Lemma 3 yield that replacing G by $\mathbf{SL}_2$ does not change $(G : G/B)$. For $G = \mathbf{SL}_2$, we have $G/B = \mathbf{P}^1$. It is classically known that the natural action of $\mathbf{PGL}_2 = \mathrm{Aut}\,\mathbf{P}^1$ on $\mathbf{P}^1$ is 3-transitive but not 4-transitive. Whence $\mathrm{gtd}(G : G/B) = 3$. $\square$

Lemma 6. *Let α be the action of a connected reductive group G on G/H, where H is a proper closed reductive subgroup of G. Then* $\mathrm{gtd}(\alpha) = 1$.

Proof. Assume the contrary. Then, since H is the G-stabilizer of a point of G/H, Lemma 4 yields that G/H contains an open H-orbit. On the other hand, by [Lu], [Ni], reductivity of H and G yields that the action of H on G/H is stable, i.e., a general H-orbit is closed in G/H. Hence the action of H on G/H is transitive. But this is impossible since the fixed point set of this action is nonempty. $\square$

Proposition 3. *Let G be a nontrivial connected reductive group. Then*

(i) $\mathrm{gtd}(G) = 1$ *if and only if G is abelian (i.e., a torus);*

(ii) *if G is nonabelian, then for some proper maximal parabolic subgroup P of G,*

$$\mathrm{gtd}(G) = \mathrm{gtd}(\alpha), \tag{9}$$

where α is the action of G on G/P.

Proof. Since solvable reductive groups are tori, (i) follows from Proposition 1.

Assume now that G is nonabelian. The same argument as in the proof of Proposition 1 shows that there a proper closed maximal subgroup H of G such that for the action α of G on G/H condition (9) holds. It is known, cf. [Hu, 30.4], that the maximality of H yields that H is either a reductive or a parabolic subgroup of G. But (i) yields $\mathrm{gtd}(G) \geqslant 2$. Therefore (9) and Lemma 6 rule out the first possibility. Thus H is a proper maximal parabolic subgroup of G. This proves (ii). $\square$

Remark 5. It would be interesting to classify subgroups Q of connected nonabelian reductive groups G such that the action of G on G/Q is generically 2-transitive. By Lemma 4, this property is equivalent to the existence of an open Q-orbit in G/Q. Lemma 6 shows that such Q is not reductive. By Lemma 5, every proper parabolic subgroup of G has this property. However the following example shows that nonparabolic subgroups with this property exist as well.

Example 2. Take a reductive group G such that the longest element w_0 of the Weyl group of G (with respect to a system of simple reflections) is not equal to $-\mathrm{id}$. Let B be a Borel subgroup of G with the unipotent radical U. Fix a maximal torus T of B and let $\dot{w}_0$ be a representative of w_0 in the normalizer of T. The above condition on w_0 yields that T contains a subtorus T' of codimension 1 that is not stable with respect to the conjugation by $\dot{w}_0$. Then $T = \dot{w}_0 T' \dot{w}_0^{-1} T'$ because of the dimension reason. Since $\dot{w}_0 B \dot{w}_0^{-1} B$ is open in G, this yields that $Q := T'U$ is a nonparabolic subgroup of G such that the action of G on G/Q is 2-transitive. The generalizations of this construction replacing 1 by a bigger codimension and B by a parabolic subgroup are clear.

Proposition 4. *Let $\gamma : \widetilde{G} \to G$ be an isogeny of nontrivial connected reductive groups. Then*

$$\mathrm{gtd}(\widetilde{G}) = \mathrm{gtd}(G).$$

Proof. If G is a torus, then $\widetilde{G}$ is a torus as well and the claim follows from Proposition 3(i). Assume now that G is nonabelian. By Proposition 3(ii), there is a parabolic subgroup P of $\widetilde{G}$ such that

$$\operatorname{gtd}(\widetilde{G}) = \operatorname{gtd}(\widetilde{\alpha}), \tag{10}$$

where $\widetilde{\alpha}$ is the action of $\widetilde{G}$ on $\widetilde{G}/P$. As γ is an isogeny, $\ker\gamma$ lies in the center of $\widetilde{G}$ that in turn lies in P, [Sp, 7.6.4]. So $\ker\gamma$ acts trivially on $\widetilde{G}/P$ and hence $\widetilde{\alpha}$ descends to the action α of G on $\widetilde{G}/P$ such that $\operatorname{gtd}(\alpha) = \operatorname{gtd}(\widetilde{\alpha})$. The claim now follows from (10), Definition 2, and Lemma 1. $\square$

Since every nonabelian connected reductive group G admits an isogeny

$$Z \times S_1 \times \ldots \times S_d \to G,$$

where Z is a torus and each S_i is a simply connected simple algebraic group, [Bor, 22.9, 22.10], [Sp, 8.1.11, 10.1.1], Propositions 3 and 4 reduce calculating generic transitivity degrees to connecting reductive groups to calculating the numbers $\operatorname{gtd}(Z \times S_1 \times \ldots \times S_d)$.

Proposition 5. *In the previous notation, there is an index j and a proper maximal parabolic subgroup P_j of S_j such that*

$$\operatorname{gtd}(Z \times S_1 \times \ldots \times S_d) = \operatorname{gtd}(\beta_j),$$

where β_j is the action of $Z \times S_1 \times \ldots \times S_d$ on S_j/P_j given by $(z, s_1, \ldots, s_d) \cdot x := s_j \cdot x$.

Proof. As all proper maximal parabolic subgroups of $Z \times S_1 \times \ldots \times S_d$ are exactly the subgroups obtained from $Z \times S_1 \times \ldots \times S_d$ by replacing some S_i with a proper maximal parabolic subgroup of S_i (see Section 4), the claim follows from Proposition 3(ii). $\square$

Corollary. *In the previous notation,*

$$\operatorname{gtd}(Z \times S_1 \times \ldots \times S_d) = \max_i \operatorname{gtd}(S_i). \quad \square$$

4. Standard parabolic subgroups

In this section we collect some necessary known facts about parabolic subgroups.

Let G be a connected reductive group. Fix a maximal torus T of G. Let Φ, Φ^+ and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be respectively the root system of G with respect to T, the system of positive roots and the system of simple roots of Φ determined by a fixed Borel subgroup containing T. For a root $\alpha \in \Phi$, let u_α be the one-dimensional unipotent root subgroup of G corresponding to α.

If I is a subset of Δ, denote by Φ_I the set of roots that are linear combinations of the roots in I. Let L_I be the subgroup of G generated by T and all the u_α's with $\alpha \in \Phi_I$. Let U_I (respectively, U_I^-) be the subgroup of G generated by all u_α with $\alpha \in \Phi^+ \setminus \Phi_I$ (respectively, $-\alpha \in \Phi^+ \setminus \Phi_I$). Then $P_I := L_I U_I$ and $P_I^- := L_I U_I^-$ are parabolic subgroups of G opposite to one another, U_I and U_I^- are the unipotent radicals of P_I and P_I^- respectively, L_I is a Levi subgroup of P_I and P_I^-. In particular,

$$\dim G = \dim L_I + 2 \dim U_I^-.$$

Every parabolic subgroup of G is conjugate to a unique P_I, called *standard*, [Sp, 8.4.3]. We denote by $\mathfrak{u}_I^-$ the Lie algebra of U_I^-. For $I = \Delta \setminus \{\alpha_i\}$, we denote L_I, U_I^-, $\mathfrak{u}_I^-$, and P_I respectively by L_i, U_i^-, $\mathfrak{u}_i^-$, and P_i. Up to conjugacy, $P_1, \ldots, P_r$ are all nonconjugate proper maximal parabolic subgroups of G.

Let w_0 be the element of the Weyl group of G with maximal length (regarding Δ), and let $\dot{w}_0$ be a representative of w_0 in the normalizer of T. Then there is an automorphism ε of Φ such that $\varepsilon(\Delta)=\Delta$ and $\dot{w}_0 P_I^- \dot{w}_0^{-1} = P_{\varepsilon(I)}$. If G is simple, then ε is given by Table 6, cf. [Bou]:

type of G	ε
A_l	$\varepsilon(\alpha_i)=\alpha_{l+1-i}$ for all i
B_l, C_l, E_7, E_8, F_4, G_2	id
D_l	id if l is even, $\left.\begin{array}{c}\varepsilon(\alpha_{l-1})=\alpha_l,\ \varepsilon(\alpha_l)=\alpha_{l-1},\\ \varepsilon(\alpha_i)=\alpha_i \text{ for } i\neq l-1,l\end{array}\right\}$ if l is odd
E_6	$\varepsilon(\alpha_1)=\alpha_6,\ \varepsilon(\alpha_2)=\alpha_2,\ \varepsilon(\alpha_3)=\alpha_5,$ $\varepsilon(\alpha_4)=\alpha_4,\ \varepsilon(\alpha_5)=\alpha_3,\ \varepsilon(\alpha_6)=\alpha_1$

Table 6

So P_I is conjugate to P_I^- if and only if $\varepsilon(I)=I$. In particular, P_i is conjugate to P_i^- if and only if $\varepsilon(\alpha_i)=\alpha_i$.

By the argument from the proof of Proposition 4, replacing G by an isogenous group does not change $\operatorname{gtd}(G : G/P_I)$. On the other hand, if G is a product of simply connected simple algebraic groups and a torus, then ε is induced by an automorphism of G stabilizing T, cf. [Hu, 32.1]. Hence, by Lemma 1(i), for every G and I, we have

$$\operatorname{gtd}(G : P_I) = \operatorname{gtd}(G : P_I^-). \tag{11}$$

The center of reductive group L_I is $(r-|I|)$-dimensional. The root system of L_I with respect to T is Φ_I. The subgroup of L_I generated by T and all the u_α's with $\alpha\in\Phi_I^+ := \Phi^+\cap\Phi_I$ is a Borel subgroup B_I of L_I. The systems of positive roots and simple roots of Φ_I determined by B_I are respectively Φ_I^+ and I. Thus the Dynkin diagram of (L_I, L_I) is obtained from that of (G,G) by removing the nodes corresponding to the elements of $\Delta\setminus I$ togeher with the adjacent edges. In the sequel, "roots", "positive roots" and "simple roots" of L_I mean the elements of Φ_I, Φ_I^+ and I respectively, and highest weights of irreducible L_I-modules are taken with respect to T and B_I.

The L-module structure of $\mathfrak{u}_I^-$ determined by the adjoint action is described as follows, [ABS] (see also [Röh]). Every such module is a direct sum of pairwise nonisomorphic irreducible submodules. Every such submodule is completely (up to isomorphism) determined by its highest weight. To describe these highest weights, it is convenient to use the following terminology and notation. Given a root $\beta\in\Phi^+\setminus\Phi_I^+$, write it as a linear combination of simple roots,

$$\beta = \textstyle\sum_{\alpha\in I} a_\alpha\alpha + \sum_{\alpha\in\Delta\setminus I} b_\alpha\alpha.$$

Call $\sum_{\alpha\in I} a_\alpha + \sum_{\alpha\in\Delta\setminus I} b_\alpha$ the *height*, $\sum_{\alpha\in\Delta\setminus I} b_\alpha\alpha$ the *shape*, and $\sum_{\alpha\in\Delta\setminus I} b_\alpha$ the *level* of β. Among all roots β having the same shape there is a unique one, β_0, whose height is minimal. Then $-\beta_0$ is the highest weight of one of the irreducible submodules of the L_I-module $\mathfrak{u}_I^-$; the shape and level of β_0 are called the shape and level of this submodule.

The action of the center of L_I on this submodule is nontrivial. The highest weight of every irreducible submodule M of the L_I-module $\mathfrak{u}_I^-$ is obtained in this way. The sum of all the M's of level i is isomorphic to the L_I-module $(\mathfrak{u}_I^-)^{(i)}/(\mathfrak{u}_I^-)^{(i+1)}$, where $(\mathfrak{u}_I^-)^{(i)}$ is the i-th term of the lower central series of $\mathfrak{u}_I^-$. By [Ri$_2$], there are only finitely many L_I-orbits in each $(\mathfrak{u}_I^-)^{(i)}/(\mathfrak{u}_I^-)^{(i+1)}$.

According to this description, for L_i, only the shapes of the form $b\alpha_i$ may occur, so α_i is the unique root of level 1 and $\mathfrak{u}_I^-/[\mathfrak{u}_I^-,\mathfrak{u}_I^-]$ is an irreducible L_i-module with the highest weight $-\alpha_i$. In particular, if $\mathfrak{u}_I^-$ is abelian, $\mathfrak{u}_I^-$ is an irreducible L_i-module with the highest weight $-\alpha_i$.

In Sections 5–13 we shall find $\operatorname{gtd}(L_i : \mathfrak{u}_i^-)$ for every connected simple algebraic group G and every $i = 1, \ldots, r$.

5. $\operatorname{gtd}(L_i : \mathfrak{u}_i^-)$ for G of type A_l

Proposition 6. *Let G be a connected simple algebraic group of type A_l. Then*

$$\operatorname{gtd}(L_i : \mathfrak{u}_i^-) = \begin{cases} 1 & \text{if } 2i = l+1, \\ s_{li} & \text{if } 2i \neq l+1 \end{cases}$$

where s_{li} is defined by formulas (4), (3).

Proof. We may (and shall) assume that

$$G = \mathbf{SL}_{l+1}, \quad P_i = \Big\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \in \mathbf{SL}_{l+1} \mid A \in \mathbf{GL}_i \Big\}, \tag{12}$$

$$L_i = \Big\{ \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \in \mathbf{SL}_{l+1} \mid A \in \mathbf{GL}_i \Big\}, \quad \mathfrak{u}_i^- = \operatorname{Mat}_{(l+1-i)\times i}.$$

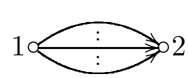
The action of L_i on $\mathfrak{u}_i^-$ is given by

$$\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \cdot X := CXA^{-1}.$$

This yields

$$\dim L_i = 2i^2 + l^2 - 2li + 2l - 2i, \qquad \dim \mathfrak{u}_i^- = il - i^2 + i \tag{13}$$

and shows that the action of L_i on $(\mathfrak{u}_i^-)^{\oplus a}$ (the direct sum of a copies of $\mathfrak{u}_i^-$) is equivalent in the sense of [SK, Definition 4, p. 36] to the action of $\mathbf{GL}_i \times \mathbf{GL}_{l+1-i}$ on the space of $(i, l+1-i)$-dimensional representations of the quiver



with a arrows. From [Ka$_1$, Theorem 4] we obtain that for $a \geqslant 2$ an open orbit in this space exists if and only if (a) $\mathcal{S}_{li} \neq \varnothing$ and (b) $a \in \mathcal{S}_{li}$. Since condition (a) is equivalent to the inequality $2i \neq l+1$, we now deduce from (4) and Definition 2 that $\operatorname{gtd}(L_i : \mathfrak{u}_i^-) = s_{li}$ if $2i \neq l+1$. If $2i = l+1$, then (13) yields $\dim L_i < 2 \dim \mathfrak{u}_i^-$, and hence in this case there is no open L_i-orbit in $(\mathfrak{u}_i^-)^{\oplus a}$ for $a \geqslant 2$. On the other hand, if $a = 1$, then, by Remark 4, such an orbit exists. $\square$

6. $\mathbf{gtd}(L_i : \mathfrak{u}_i^-)$ for G of type B_l, $l \geqslant 3$

Proposition 7. *Let G be a connected simple algebraic group of type B_l, $l \geqslant 3$. Then*

$$\operatorname{gtd}(L_i : \mathfrak{u}_i^-) = \begin{cases} 0 & \text{if } i \neq 1, l, \\ 1 & \text{if } i = 1, l. \end{cases}$$

Proof. Step 1. Let $i = 1$. By Remark 4 and Corollary 2 of Proposition 2, the action of L_1 on $\mathfrak{u}_1^-$ is locally transitive. The type of (L_1, L_1) is B_{l-1}. The action of L_1 on $\mathfrak{u}_1^-$ is irreducible with the highest weight $-\alpha_1 = (-2, 1, 0, \ldots, 0)$. Hence $\mathfrak{u}_1^-$, considered as (irreducible) (L_1, L_1)-module, has the highest weight with the Dynkin diagram

$$\overset{1}{\circ}\!-\!\cdots\!-\!\circ\!\Rightarrow\!\circ \quad . \tag{14}$$

From (14) we deduce that, for a general point $z \in \mathfrak{u}_1^-$, the group $(L_1, L_1)_z^0$ is locally isomorphic to $\mathbf{SO}_{2l-1}$, and the codimension of $(L_1, L_1) \cdot z$ in $\mathfrak{u}_1^-$ is equal to 1. As the (one-dimensional) center of L_1 acts on $\mathfrak{u}_1^-$ nontrivially and the action of L_1 on $\mathfrak{u}_1^-$ is locally transitive, this yields $(L_1)_z^0 = (L_1, L_1)_z^0$. Hence the action of $(L_1)_z^0$ on $\mathfrak{u}_1^-$ is not locally transitive. Lemma 4 now yields $\operatorname{gtd}(L_1 : \mathfrak{u}_1^-) = 1$.

Step 2. Let $i = l$. The type of (L_l, L_l) is A_{l-1}. By inspection of Φ^+ in [Bou, Planche II] we obtain that, for the action of L_l on $\mathfrak{u}_l^-$, there are exactly two shapes α_l and $2\alpha_l$, and they determine the highest weights $-\alpha_l = (0, \ldots, 0, 1, -2)$ and $-\alpha_{l-1} - 2\alpha_l = (0, \ldots, 0, 1, 0, -2)$. Hence $\mathfrak{u}_1^-$ is the direct sum of two irreducible L_l-modules $\mathfrak{u}_{l1}^-$ and $\mathfrak{u}_{l2}^-$ that, considered as (irreducible) (L_l, L_l)-modules, have respectively the highest weights with the Dynkin diagrams

$$\circ\!-\!\cdots\!-\!\overset{1}{\circ} \qquad \text{and} \qquad \circ\!-\!\cdots\!-\!\overset{1}{\circ}\!-\!\circ \quad . \tag{15}$$

From (15) and [Kimu, Section 3, A(8)(iii) and B(4)(iii)] we conclude that the action of L_l on $\mathfrak{u}_l^-$ is locally transitive. On the other hand, if the action of L_l on $\mathfrak{u}_l^- \oplus \mathfrak{u}_l^-$ would be locally transitive, then all the more the action of $\mathbf{GL}_1^4 \times (L_l, L_l)$ on $\mathfrak{u}_{l1}^- \oplus \mathfrak{u}_{l1}^- \oplus \mathfrak{u}_{l2}^- \oplus \mathfrak{u}_{l2}^-$, where $\mathbf{GL}_1^4$ acts on the direct summands by independent scalar multiplications, would be locally transitive. Using the terminology and notation of [Kimu], this would mean that the pair $(\mathbf{GL}_1^4 \times \mathbf{SL}_l, \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_2)$ is prehomogeneous. However the classification obtained in [Kimu, Section 3] shows that it is not so. Thus $\operatorname{gtd}(L_l : \mathfrak{u}_l^-) = 1$.

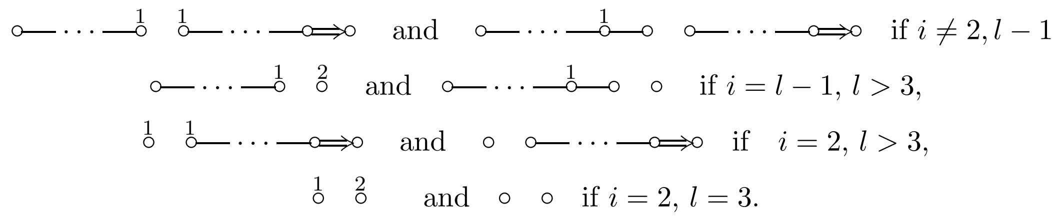
Step 3. The type of (L_i, L_i) for $1 < i < l$ is $\mathsf{A}_{i-1} + \mathsf{B}_{l-i}$, where $\mathsf{B}_1 := \mathsf{A}_1$. By inspection of Φ^+ in [Bou, Planche II] we obtain that for the action of L_i on $\mathfrak{u}_i^-$ there are exactly two shapes α_i and $2\alpha_i$, and they determine respectively the highest weights

$$-\alpha_i = \begin{cases} (0, \ldots, 0, 1, \underset{\hat{i}}{-2}, 1, 0, \ldots, 0) & \text{if } i \neq l-1, \\ (0, \ldots, 0, 1, -2, 2) & \text{if } i = l-1, \end{cases} \quad \text{and}$$

$$-\alpha_{i-1} - 2\alpha_i - \ldots - 2\alpha_l = \begin{cases} (0, \ldots, 0, 1, 0, \underset{\hat{i}}{-1}, 0, \ldots, 0) & \text{if } i \neq 2, \\ (0, -1, 0, \ldots, 0) & \text{if } i = 2. \end{cases}$$

Hence $\mathfrak{u}_i^-$ is the direct sum of two irreducible L_i-modules $\mathfrak{u}_{i1}^-$ and $\mathfrak{u}_{i2}^-$ that, considered as (irreducible) (L_i, L_i)-modules, have respectively the highest weights with the Dynkin

diagrams



(16)

Step 4. Let $i = 2$. From (16) we deduce that $\dim \mathfrak{u}_{22}^- = 1$. Since the action of L_2 on $\mathfrak{u}_{22}^-$ is locally transitive and the center of L_2 is one-dimensional, $(L_2)_z^0 = (L_2, L_2)$ for a nonzero $z \in \mathfrak{u}_{22}^-$. This and (16) now yield that, in the notation and terminology of [SK], [KKIY], [KKTI], the action of $(L_2)_z^0$ on $\mathfrak{u}_{21}^-$ is equivalent to that determined by the pair $(\mathbf{SL}_2 \times \mathbf{SO}_{2l-3}, \Lambda_1 \otimes \Lambda_1)$. By [SK, § 6], this pair is not prehomogeneous. So the action of $(L_2)_z^0$ on $\mathfrak{u}_{21}^-$ is not locally transitive. Hence, by Lemma 4, the action of L_2 on $\mathfrak{u}_2^-$ is not locally transitive as well.

Step 5. Let $i = l-1$ and $i \neq 2$ (hence $l > 3$). From (16) we deduce that the action of L_{l-1} on $\mathfrak{u}_{l-1}^-$ is equivalent to that determined by the pair $(\mathbf{GL}_{l-1} \times \mathbf{SO}_3, \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1)$. This shows that the action of L_{l-1} on $\mathfrak{u}_{l-1,2}^-$ is equivalent to the natural action of $\mathbf{GL}_{l-1}$ on the space of skew-symmetric bilinear forms over k in $l-1$ variables.

Assume that l is odd. The last remark then yields that, for a general point $z \in \mathfrak{u}_{l-1,2}^-$, the group $(L_{l-1})_z^0$ is locally isomorphic to $\mathbf{Sp}_{l-1} \times \mathbf{SO}_3$ and its action on $\mathfrak{u}_{l-1,1}^-$ is equivalent to that determined by the pair $(\mathbf{Sp}_{l-1} \times \mathbf{SO}_3, \Lambda_1 \otimes \Lambda_1)$. By [SK, § 6], this pair is not prehomogeneous. So the action of $(L_{l-1})_z^0$ on $\mathfrak{u}_{l-1,1}^-$ is not locally transitive. Hence, by Lemma 4, if l is odd, then the action of L_{l-1} on $\mathfrak{u}_{l-1}^-$ is not locally transitive.

Assume now that l is even. Then, according to [KKTI, Proposition 1.34 (2)], the prehomogeneity of $(\mathbf{GL}_{l-1} \times \mathbf{SO}_3, \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1)$ is equivalent to that of $(\mathbf{GL}_2 \times \mathbf{SO}_3, \Lambda_1 \otimes \Lambda_1 + \det \otimes 1)$. In turn, since for $(\mathbf{GL}_2 \times \mathbf{SO}_3, \det \otimes 1)$ the stabilizer of a general point is $\mathbf{SL}_2 \times \mathbf{SO}_3$, Lemma 4 shows that the prehomogeneity of $(\mathbf{GL}_2 \times \mathbf{SO}_3, \Lambda_1 \otimes \Lambda_1 + \det \otimes 1)$ is equivalent to that of $(\mathbf{SL}_2 \times \mathbf{SO}_3, \Lambda_1 \otimes \Lambda_1)$. But according to [SK, § 6], the last pair is not prehomogeneous.

Summing up, we obtain that for every l the action of L_{l-1} on $\mathfrak{u}_{l-1}^-$ is not locally transitive.

Step 6. Let $2 < i < l-1$. From (16) we deduce that the action of L_i on $\mathfrak{u}_i^-$ is equivalent to that determined by the pair $(\mathbf{GL}_i \times \mathbf{SO}_{2(l-i)+1}, \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1)$. If the action of L_i on $\mathfrak{u}_i^-$ would be locally transitive, then all the more the action determined by the pair $(\mathbf{GL}_1^2 \times \mathbf{SL}_i \times \mathbf{SO}_{2(l-i)+1}, \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1)$, where $\mathbf{GL}_1^2$ acts on the summands by independent scalar multiplications, would be locally transitive. In turn, this would mean that if $2(l-i)+1 > i$, then the last pair is 2-simple prehomogeneous of type I in the sense of [KKIY]. However the classification of such pairs obtained in [KKIY, Section 3] shows that is is not so. Hence, for $2(l-i)+1 > i$, the action of L_i on $\mathfrak{u}_i^-$ is not locally transitive.

Assume now that $2(l-i)+1 \leqslant i$. Then, by the same argument, if the action of L_i on $\mathfrak{u}_i^-$ would be locally transitive, then the pair $(\mathbf{GL}_1^2 \times \mathbf{SL}_i \times \mathbf{SO}_{2(l-i)+1}, \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1)$ would be 2-simple prehomogeneous of type II in the sense of [KKIY], [KKTI]. However the classification of such pairs obtained in [KKTI, Section 5] shows that it is not so. Hence if $2(l-i)+1 \leqslant i$, then the action of L_i on $\mathfrak{u}_i^-$ is not locally transitive.

Summing up, we obtain that if $2 < i < l-2$, then the action of L_i on $\mathfrak{u}_i^-$ is not locally transitive. $\square$

Remark 6. In [Kime$_1$], [Kime$_2$], for every i, it is obtained a classification of all connected reductive subgroups of $G = \mathbf{SO}_n(\mathbb{C})$ that act locally transitively on G/P_i. $\square$

7. $\mathrm{gtd}(L_i : \mathfrak{u}_i^-)$ for G of type C_l, $l \geqslant 2$

Proposition 8. *Let G be a connected simple algebraic group of type C_l, $l \geqslant 2$. Then*

$$\mathrm{gtd}(L_i : \mathfrak{u}_i^-) = \begin{cases} 0 & \text{if } i \neq 1, l, \\ 1 & \text{if } i = 1, l. \end{cases}$$

Proof. Step 1. Let $i = 1$. The type of (L_1, L_1) is C_{l-1}, where $\mathsf{C}_1 := \mathsf{A}_1$. By inspection of Φ^+ in [Bou, Planche III] we obtain that, for the action of L_1 on $\mathfrak{u}_1^-$, there are exactly two shapes α_1 and $2\alpha_1$, and they determine the highest weights $-\alpha_1 = (-2, 1, 0, \ldots, 0)$ and $-2\alpha_1 - \ldots - 2\alpha_{l-1} - \alpha_l = (-2, 0, \ldots, 0)$. Hence $\mathfrak{u}_1^-$ is the direct sum of two irreducible L_1-modules $\mathfrak{u}_{11}^-$ and $\mathfrak{u}_{12}^-$, where $\mathfrak{u}_{11}^-$, considered as (irreducible) (L_1, L_1)-module, has the highest weight with the Dynkin diagram

$$\begin{array}{ll} \overset{1}{\circ}\!-\!\cdots\!-\!\circ\!\Leftarrow\!\circ & \text{if } l > 2, \\ \overset{1}{\circ} & \text{if } l = 2, \end{array} \tag{17}$$

and $\mathfrak{u}_{12}^-$ is a trivial one-dimensional module.

As the action of L_1 on $\mathfrak{u}_{12}^-$ is locally transitive and (L_1, L_1) has no nontrivial characters, $(L_1)_z^0 = (L_1, L_1)$ for a nonzero point $z \in \mathfrak{u}_{12}^-$. It follows from (17) that the action of (L_1, L_1) on $\mathfrak{u}_{11}^-$ is equivalent, in the sense of [SK, Definition 4, p. 36], to the natural action of $\mathbf{Sp}_{2l-2}$ on k^{2l-2}. As the latter is locally transitive by Witt's theorem, Lemma 4 yields that the action of L_1 on $\mathfrak{u}_1^-$ is locally transitive. Since $\mathbf{Sp}_{2l-2}$ fixes a nondegenerate skew-symmetric form on k^{2l-2}, the natural action of $\mathbf{Sp}_{2l-2}$ on $k^{2l-2} \oplus k^{2l-2}$ is not locally transitive. Applying the same argument as above we then conclude that $\mathrm{gtd}(L_1 : \mathfrak{u}_1^-) = 1$.

Step 2. Let $i = l$. By Remark 4 and Corollary 2 of Proposition 2, the action of L_l on $\mathfrak{u}_l^-$ is locally transitive. The type of (L_l, L_l) is A_{l-1}, so $\dim L_l = l^2$, $\dim \mathfrak{u}_l^- = l(l+1)/2$. As $\dim L_l < 2 \dim \mathfrak{u}_l^-$, we have $\mathrm{gtd}(L_l : \mathfrak{u}_l^-) = 1$.

Step 3. Let $l \geqslant 3$ and $1 < i < l$. The type of (L_i, L_i) is $\mathsf{A}_{i-1} + \mathsf{C}_{l-i}$. By inspection of Φ^+ in [Bou, Planche III] we obtain that, for the action of L_i on $\mathfrak{u}_i^-$, there are exactly two shapes α_i and $2\alpha_i$, and they determine respectively the highest weights $-\alpha_i = (0, \ldots, 0, 1, \underset{\hat{i}}{-2}, 1, 0, \ldots, 0)$ and $-2\alpha_i - \ldots - 2\alpha_{l-1} - \alpha_l = (0, \ldots, 0, 2, \underset{\hat{i}}{-2}, 0 \ldots, 0)$. Hence $\mathfrak{u}_i^-$ is the direct sum of two irreducible L_i-modules $\mathfrak{u}_{i1}^-$ and $\mathfrak{u}_{i2}^-$ that, considered as (irreducible) (L_i, L_i)-modules, have respectively the highest weights with the Dynkin diagrams

$$\begin{array}{lll} \circ\!-\!\cdots\!-\!\overset{1}{\circ}\quad \overset{1}{\circ}\!-\!\cdots\!-\!\circ\!\Leftarrow\!\circ \quad \text{and} \quad \circ\!-\!\cdots\!-\!\overset{2}{\circ}\quad \circ\!-\!\cdots\!-\!\circ\!\Leftarrow\!\circ & \text{if } i \neq l-1, \\ \circ\!-\!\cdots\!-\!\overset{1}{\circ}\quad \overset{1}{\circ} \quad \text{and} \quad \circ\!-\!\cdots\!-\!\overset{2}{\circ}\quad \circ & \text{if } i = l-1, \end{array} \tag{18}$$

From (18) we deduce that $\dim \mathfrak{u}_{i2}^- = i(i+1)/2$, for a general point $z \in \mathfrak{u}_{i2}^-$ the group $(L_i, L_i)_z^0$ is locally isomorphic to $\mathbf{SO}_i \times \mathbf{Sp}_{2l-2i}$, and the codimension of orbit $(L_i, L_i) \cdot z$ in $\mathfrak{u}_{i2}^-$ is equal to 1. As the center of L_i is one-dimensional and acts on $\mathfrak{u}_{i2}^-$ nontrivially, and the

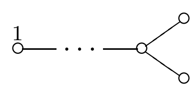
action of L_i on $\mathfrak{u}_{i2}^-$ is locally transitive, this yields $(L_i)_z^0 = (L_i, L_i)_z^0$. So $(L_i)_z^0$ is locally isomorphic to $\mathbf{SO}_i \times \mathbf{Sp}_{2l-2i}$. From (18) we now deduce that, in the terminology and notation of [SK, Definition 4, p. 36], the action of $(L_i)_z^0$ on $\mathfrak{u}_{i1}^-$ is equivalent to that determined by the pair $(\mathbf{SO}_i \times \mathbf{Sp}_{2l-2i}, \Lambda_1 \otimes \Lambda_1)$. By [SK, § 7], this pair is not prehomogeneous. Lemma 4 now yields that the action of L_i on $\mathfrak{u}_i^-$ is not locally transitive. $\square$

8. $\mathbf{gtd}(\boldsymbol{L_i} : \boldsymbol{\mathfrak{u}_i^-})$ for $\boldsymbol{G}$ of type $\mathsf{D}_{\boldsymbol{l}}$, $\boldsymbol{l \geqslant 4}$

Proposition 9. *Let G be a connected simple algebraic group of type D_l, $l \geqslant 4$. Then*

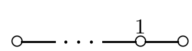
$$\operatorname{gtd}(L_i : \mathfrak{u}_i^-) = \begin{cases} 0 & \text{if } i \neq 1, l-1, l, \\ 1 & \text{if } i = 1, \\ 1 & \text{if } l \text{ is even and } i = l-1, l, \\ 2 & \text{if } l \text{ is odd and } i = l-1, l. \end{cases}$$

Proof. Step 1. Let $i = 1$. By Remark 4 and Corollary 2 of Proposition 2, the action of L_1 on $\mathfrak{u}_1^-$ is locally transitive. The type of (L_1, L_1) is D_{l-1}, where $\mathsf{D}_3 := \mathsf{A}_3$. The action of L_1 on $\mathfrak{u}_1^-$ is irreducible with the highest weight $-\alpha_1 = (-2, 1, 0, \ldots, 0)$. Hence $\mathfrak{u}_1^-$, considered as (irreducible) (L_1, L_1)-module, has the highest weight with the Dynkin diagram

 . (19)

Arguing like in Subsection 7 for $l \geqslant 3$, we deduce from (19) that, for a general point $z \in \mathfrak{u}_i^-$, the group $(L_1)_z^0$ lies in (L_1, L_1). As the action of (L_1, L_1) on $\mathfrak{u}_1^-$ is not locally transitive (it fixes a nondegenerate quadratic form), Lemma 4 now yields $\operatorname{gtd}(L_1 : \mathfrak{u}_1^-) = 1$.

Step 2. Let $i = l-1, l$. Again by Remark 4 and Corollary 2 of Proposition 2, the action of L_l on $\mathfrak{u}_l^-$ is locally transitive. The type of (L_l, L_l) is A_{l-1}. The action of L_l on $\mathfrak{u}_l^-$ is irreducible with the highest weight $-\alpha_l = (0, \ldots, 0, 1, 0, -2)$. Hence $\mathfrak{u}_l^-$, considered as (irreducible) (L_l, L_l)-module, has the highest weight with the Dynkin diagram

 . (20)

From (20) we deduce that, in the terminology and notation of [SK, Definition 4, p. 36], the action of (L_l, L_l) on $\mathfrak{u}_l^- \oplus \mathfrak{u}_l^-$ is equivalent to that determined by the pair $(\mathbf{SL}_l, \Lambda_2 \oplus \Lambda_2)$. Since the center of L_l is one-dimensional, it now follows from [Kimu, Proposition 2.2 and Section 3, B (3)(iii)] that the action of L_l on $\mathfrak{u}_l^- \oplus \mathfrak{u}_l^-$ is not locally transitive for even l, and is locally transitive for odd l. On the other hand, as $\dim L_l = l^2 < 3l(l-1)/2 = \dim(\mathfrak{u}_l^- \oplus \mathfrak{u}_l^- \oplus \mathfrak{u}_l^-)$, the action of L_l on $\mathfrak{u}_l^- \oplus \mathfrak{u}_l^- \oplus \mathfrak{u}_l^-$ is not locally transitive. So we see that $\operatorname{gtd}(L_l : \mathfrak{u}_l^-)$ is equal to 1 if l is even, and to 2 if l is odd.

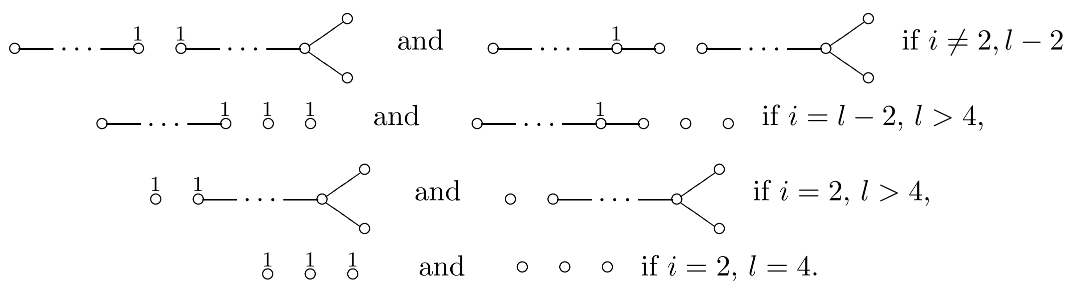
By Proposition 2 and (11), we have $\operatorname{gtd}(L_{l-1} : \mathfrak{u}_{l-1}^-) = \operatorname{gtd}(L_l : \mathfrak{u}_l^-)$.

Step 3. If $1 < i \leqslant l-2$, then the type of (L_i, L_i) is $\mathsf{A}_{i-1} + \mathsf{D}_{l-i}$, where $\mathsf{D}_2 := \mathsf{A}_1 + \mathsf{A}_1$. By inspection of [Bou, Planche IV] we obtain that for the action of L_i on $\mathfrak{u}_i^-$ there are exactly two shapes α_i and $2\alpha_i$, and they determine respectively the highest weights

$$-\alpha_i = \begin{cases} (0, \ldots, 1, \underset{\hat{i}}{-2}, 1, 0, \ldots, 0) & \text{if } i \neq l-2, \\ (0, \ldots, 0, 1, -2, 1, 1) & \text{if } i = l-2, \end{cases} \quad \text{and}$$

$$-\alpha_{i-1} - 2\alpha_i - \ldots - 2\alpha_{l-2} - \alpha_{l-1} - \alpha_l = \begin{cases} (0,\ldots,0,1,0,\underset{\hat{i}}{-1},0,\ldots,0) & \text{if } i \neq 2,\\ (0,-1,0,\ldots,0) & \text{if } i = 2. \end{cases}$$

Hence $\mathfrak{u}_i^-$ is the direct sum of two irreducible L_i-modules $\mathfrak{u}_{i1}^-$ and $\mathfrak{u}_{i2}^-$ that, considered as (irreducible) (L_i, L_i)-modules, have respectively the highest weights with the Dynkin diagrams



(21)

Step 4. Let $i = 2$. From (21) we deduce that $\dim \mathfrak{u}_{22}^- = 1$ and the action of (L_2, L_2) on $\mathfrak{u}_{21}^-$ is equivalent to that determined by the pair $(\mathbf{SL}_2 \times \mathbf{SO}_{2(l-1)}, \Lambda_1 \otimes \Lambda_1)$ for $l > 4$, and to the pair $(\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ for $l = 4$. By [SK, § 6], these pairs are not prehomogeneous. Using now the same argument as in the case $i = 2$ in Subsection 6 we obtain that the action of L_2 on $\mathfrak{u}_2^-$ is not locally transitive.

Step 5. Let $i = l-2$ and $i \neq 2$ (hence $l > 4$). From (21) we deduce that the action of L_{l-2} on $\mathfrak{u}_{l-2}^-$ is equivalent to that determined by the pair $(\mathbf{GL}_{l-2} \times \mathbf{SL}_2 \times \mathbf{SL}_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1 \otimes 1)$. Hence the action of L_{l-2} on $\mathfrak{u}_{l-2,2}^-$ is equivalent to the natural action of $\mathbf{GL}_{l-2}$ on the space of skew-symmetric bilinear forms over k in $l-2$ variables.

Assume that l is even. Then the last remark yields that, for a general point $z \in \mathfrak{u}_{l-2,2}^-$, the group $(L_{l-2})_z^0$ is locally isomorphic to $\mathbf{Sp}_{l-2} \times \mathbf{SL}_2 \times \mathbf{SL}_2$ and its action on $\mathfrak{u}_{l-2,1}^-$ is equivalent to that determined by the pair $(\mathbf{Sp}_{l-2} \times \mathbf{SL}_2 \times \mathbf{SL}_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$. As, by [SK, § 6], this pair is not prehomogeneous, we conclude that if l is even, then the action of L_{l-2} on $\mathfrak{u}_{l-2}^-$ is not locally transitive.

Assume now that l is odd. Then, according to [KKTI, Proposition 1.34 (2)], the prehomogeneity of $(\mathbf{GL}_{l-2} \times \mathbf{SL}_2 \times \mathbf{SL}_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1 \otimes 1)$ is equivalent to that of $(\mathbf{GL}_3 \times \mathbf{SL}_2 \times \mathbf{SL}_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1 \otimes 1)$. The action determined by $(\mathbf{GL}_3 \times \mathbf{SL}_2 \times \mathbf{SL}_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ is equivalent to that of $\mathbf{GL}_3 \times \mathbf{SO}_4$ on $\mathrm{Mat}_{3\times 4}$ given by

$$g \cdot X := AX^tB, \quad g = (A, B) \in \mathbf{GL}_3 \times \mathbf{SO}_4,\ X \in \mathrm{Mat}_{3\times 4}.$$

By [SK, p. 109], the $\mathbf{GL}_3 \times \mathbf{SO}_4$-orbit of the matrix $[I_3\, 0]$ is open in $\mathrm{Mat}_{3\times 4}$ and its stabilizer is

$$\left\{\left(A, \begin{bmatrix} {}^tA^{-1} & 0 \\ 0 & a \end{bmatrix}\right) \mid A \in \mathbf{O}_3,\ a = \pm 1,\ a \det A = 1\right\}.$$

Hence the action of the identity component of this stabilizer on the second summand of the pair $(\mathbf{GL}_3 \times \mathbf{SL}_2 \times \mathbf{SL}_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1 \otimes 1)$ is equivalent to the action of $\mathbf{SO}_3$ determined by the second exterior power of its natural 3-dimensional representation. Since the last action is clearly not locally transitive, Lemma 4 yields that for odd l the action of L_{l-2} on $\mathfrak{u}_{l-2}^-$ is not locally transitive.

Summing up, we obtain that for every l the action of L_{l-2} on $\mathfrak{u}_{l-2}^-$ is not locally transitive.

Step 6. Let $2 < i < l-2$. From (21) we deduce that the action of L_i on $\mathfrak{u}_i^-$ is equivalent to that determined by the pair $(\mathbf{GL}_i \times \mathbf{SO}_{2(l-i)}, \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1)$. If the action of L_i on $\mathfrak{u}_i^-$ would be locally transitive, then all the more the action determined by the pair $(\mathbf{GL}_1^2 \times \mathbf{SL}_i \times \mathbf{SO}_{2(l-i)}, \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1)$, where $\mathbf{GL}_1^2$ acts on the summands by independent scalar multiplications, would be locally transitive. In turn, this would mean that if $2(l-i) > i$, then the last pair is 2-simple prehomogeneous of type I in the sense of [KKIY]. However the classification of such pairs obtained in [KKIY, Section 3] shows that it is not so. Hence, for $2(l-i) > i$, the action of L_i on $\mathfrak{u}_i^-$ is not locally transitive.

Assume now that $2(l-i) \leqslant i$. Then, by the same argument, if the action of L_i on $\mathfrak{u}_i^-$ would be locally transitive, then the pair $(\mathbf{GL}_1^2 \times \mathbf{SL}_i \times \mathbf{SO}_{2(l-i)}, \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1)$ would be 2-simple prehomogeneous of type II in the sense of [KKIY], [KKTI]. However the classification of such pairs obtained in [KKTI, Section 5] shows that it is not so. Hence if $2(l-i) \leqslant i$, then the action of L_i on $\mathfrak{u}_i^-$ is not locally transitive.

Summing up, we obtain that if $2 < i < l-2$, then the action of L_i on $\mathfrak{u}_i^-$ is not locally transitive. $\square$

9. $\mathrm{gtd}(L_i : \mathfrak{u}_i^-)$ for G of type E_6

Proposition 10. *Let G be a connected simple algebraic group of type E_6. Then*

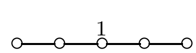
$$\mathrm{gtd}(L_i : \mathfrak{u}_i^-) = \begin{cases} 0 & \text{if } i = 2, 4, \\ 1 & \text{if } i = 3, 5, \\ 2 & \text{if } i = 1, 6. \end{cases}$$

Proof. Step 1. Let $i = 4$. We have $\dim G = 78$. The type of (L_4, L_4) is $\mathsf{A}_1 + \mathsf{A}_2 + \mathsf{A}_2$, so $\dim L_4 = 20$, $\dim \mathfrak{u}_4^- = 29$. As $\dim L_4 < \dim \mathfrak{u}_4^-$, the action of L_4 on $\mathfrak{u}_4^-$ is not locally transitive.

Step 2. Let $i = 1, 6$. By Remark 4 and Corollary 2 of Proposition 2, the action of L_1 on $\mathfrak{u}_1^-$ locally transitive. The type of (L_1, L_1) is D_5, so $\dim L_1 = 46$, $\dim \mathfrak{u}_1^- = 16$. As $\mathfrak{u}_1^-$ is abelian, the action of L_1 on $\mathfrak{u}_1^-$ is irreducible with the highest weight $-\alpha_1 = (-2, 0, 1, 0, 0, 0)$. Hence $\mathfrak{u}_1^-$ is a half-spinor module of (L_1, L_1). From [Kimu, Section 3, A, (17), (iii) and Proposition 2.23], [SK, Proposition 32] it now follows that $\mathrm{gtd}(L_1 : \mathfrak{u}_1^-) = 2$.

By Proposition 2 and (11), we have $\mathrm{gtd}(L_6 : \mathfrak{u}_6^-) = 2$.

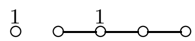
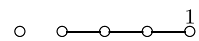
Step 3. Let $i = 2$. The type of (L_2, L_2) is A_5, so $\dim L_2 = 36$, $\dim \mathfrak{u}_2^- = 21$. By inspection of Φ^+ in [Bou, Planche V] we obtain that, for the action of L_2 on $\mathfrak{u}_2^-$, there are exactly two shapes α_2 and $2\alpha_2$, and they determine the highest roots $-\alpha_2 = (0, -2, 0, 1, 0, 0)$ and $-\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6 = (0, -1, 0, 0, 0, 0)$. Hence $\mathfrak{u}_2^-$ is the direct sum of two irreducible L_2-modules $\mathfrak{u}_{21}^-$ and $\mathfrak{u}_{22}^-$, where $\mathfrak{u}_{21}^-$, considered as (irreducible) (L_2, L_2)-module, has the highest weight with the Dynkin diagram



and $\mathfrak{u}_{22}^-$ is a trivial one-dimensional module. As the action of L_2 on $\mathfrak{u}_{22}^-$ is locally transitive, $(L_2)_z^0 = (L_2, L_2)$ for a nonzero point $z \in \mathfrak{u}_{22}^-$. Since the action of (L_2, L_2) on $\mathfrak{u}_{21}^-$ is not locally transitive, [SK, §7], Lemma 4 yields that the action of L_2 on $\mathfrak{u}_2^-$ is not locally transitive.

Step 4. Let $i = 3, 5$. The type of (L_3, L_3) is $\mathsf{A}_1 + \mathsf{A}_4$, so $\dim L_3 = 28$, $\dim \mathfrak{u}_3^- = 25$. By inspection of Φ^+ in [Bou, Planche V] we obtain that, for the action of L_3 on $\mathfrak{u}_3^-$, there are exactly two shapes α_3 and $2\alpha_3$, and they determine the highest roots $-\alpha_3 = (1, 0, -2, 1, 0, 0)$

and $-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5=(0,0,-1,0,0,1)$. Hence $\mathfrak{u}_3^-$ is the direct sum of two irreducible L_3-modules $\mathfrak{u}_{31}^-$ and $\mathfrak{u}_{32}^-$ that, considered as (irreducible) (L_3,L_3)-modules, have respectively the highest weights with the Dynkin diagrams

 and  .

So we may (and shall) identify the Lie algebra of (L_3,L_3) with the Lie algebra of matrices

$$\left\{\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix} \mid A\in \mathrm{Mat}_{2\times 2},\ B\in \mathrm{Mat}_{5\times 5},\ \mathrm{tr}A=\mathrm{tr}B=0\right\} \tag{22}$$

(the Lie bracket is given by the commutator) and $\mathfrak{u}_{32}^-$ with the coordinate space k^5 on which this Lie algebra acts by the rule

$$\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}\cdot v := -{}^t\!Bv.$$

Then, by [KKIY, Lemma 1.4], the open L_3-orbit in $\mathfrak{u}_{31}^-$ contains a point z such that

$$\left\{\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix} \mid A=\begin{bmatrix}a_1 & a_2\\ a_3 & -a_1\end{bmatrix}, B=\begin{bmatrix}C & 0\\ D & A\end{bmatrix}, C=-\begin{bmatrix}2a_1 & 2a_3 & 0\\ a_2 & 0 & a_3\\ 0 & 2a_2 & -2a_1\end{bmatrix}, D=\begin{bmatrix}a_4 & a_5 & a_6\\ a_5 & a_6 & a_7\end{bmatrix}, a_i\in k\right\}$$

is the Lie algebra of $(L_3,L_3)_z$. Hence $\dim(L_3,L_3)_z=7$ and the Lie algebra of the $(L_3,L_3)_z$-stabilizer of the point $v:={}^t(1,0,0,0,0)\in\mathfrak{u}_{32}^-$ consists of all $\left[\begin{smallmatrix}A & 0\\ 0 & B\end{smallmatrix}\right]$ with $a_1=a_3=0$. This shows that the $(L_3,L_3)_z$-orbit of v is 5-dimensional and hence open in $\mathfrak{u}_{32}^-$. All the more the action of $(L_3)_z$ on $\mathfrak{u}_{32}^-$ is locally transitive. Lemma 4 now yields that the action of L_3 on $\mathfrak{u}_3^-$ is locally transitive. As $\dim L_3<2\dim\mathfrak{u}_3^-$, this yields $\mathrm{gtd}(L_3:\mathfrak{u}_3^-)=1$.

By Proposition 2 and (11), we have $\mathrm{ltd}(L_5:\mathfrak{u}_5^-)=1$. $\square$

10. $\mathbf{gtd}(L_i:\mathfrak{u}_i^-)$ for G of type E_7

Proposition 11. *Let G be a connected simple algebraic group of type E_7. Then*

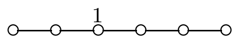
$$\mathrm{gtd}(L_i:\mathfrak{u}_i^-)=\begin{cases}0 & \text{if } i\neq 7,\\ 1 & \text{if } i=7.\end{cases}$$

Proof. Step 1. We have $\dim G=133$. Let $i=3,4,5$. Then the type of (L_i,L_i) is respectively $\mathsf{A}_1+\mathsf{A}_5$, $\mathsf{A}_1+\mathsf{A}_2+\mathsf{A}_3$, $\mathsf{A}_2+\mathsf{A}_4$. Hence respectively $\dim L_i=39,27,33$ and $\dim\mathfrak{u}_i^-=47,53,50$. As $\dim L_i<\dim\mathfrak{u}_i^-$, the action of L_i on $\mathfrak{u}_i^-$ is not locally transitive.

Step 2. Let $i=1$. The type of (L_1,L_1) is D_6, so $\dim L_1=67$, $\dim\mathfrak{u}_1^-=33$. By inspection of Φ^+ in [Bou, Planche VI] we obtain that, for the action of L_1 on $\mathfrak{u}_1^-$, there are exactly two shapes α_1 and $2\alpha_1$, and they determine the highest weights $-\alpha_1=(-2,0,1,0,0,0,0)$ and $-2\alpha_1-2\alpha_2-3\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-\alpha_7=(-1,0,0,0,0,0,0)$. Hence $\mathfrak{u}_1^-$ is the direct sum of two irreducible L_1-modules $\mathfrak{u}_{11}^-$ and $\mathfrak{u}_{12}^-$ that, considered as (irreducible) (L_1,L_1)-modules, are respectively a half-spinor and a trivial 1-dimensional module. As the action of L_1 on $\mathfrak{u}_{12}^-$ is locally transitive, $(L_1)_z^0=(L_1,L_1)$ for a general point $z\in\mathfrak{u}_{12}^-$. Since the action of (L_1,L_1) on $\mathfrak{u}_{11}^-$ is not locally transitive, [SK, § 7], Lemma 4 yields that the action of L_1 on $\mathfrak{u}_1^-$ is not locally transitive.

Step 3. Let $i=2$. The type of (L_2,L_2) is A_6, so $\dim L_2=49$, $\dim\mathfrak{u}_2^-=42$. By inspection of Φ^+ in [Bou, Planche VI] we obtain that, for the action of L_2 on $\mathfrak{u}_2^-$, there are exactly two shapes α_2 and $2\alpha_2$, and they determine the highest weights $-\alpha_2=(0,-2,0,1,0,0,0)$

and $-\alpha_1-2\alpha_2-2\alpha_3-3\alpha_4-2\alpha_5-\alpha_6=(0,-1,0,0,0,0,1)$. Hence $\mathfrak{u}_2^-$ is the direct sum of two irreducible L_2-modules $\mathfrak{u}_{21}^-$ and $\mathfrak{u}_{22}^-$ that, considered as (irreducible) (L_2,L_2)-modules, have respectively the highest weights with the Dynkin diagrams



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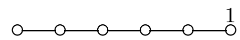

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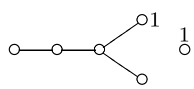
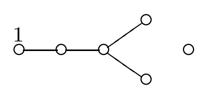
By [SK, § 7, I, (6)], if z is a general point of $\mathfrak{u}_{21}^-$, then $(L_2)_z^0$ is a simple algebraic group of type G_2. Hence $(L_2)_z^0 \subset (L_2,L_2)$ and, as (L_2,L_2) is simple, the action of $(L_2)_z^0$ on $\mathfrak{u}_{22}^-$ is nontrivial. Since the dimension of every nontrivial module of a simple group of type G_2 is at least $7=\dim\mathfrak{u}_{22}^-$, and, by [SK, § 7], the action of $\mathbf{G}_2$ on every such module is not locally transitive, this yields that the action of $(L_2)_z^0$ on $\mathfrak{u}_{22}^-$ is not locally transitive. From Lemma 4 we then deduce that the action of L_2 on $\mathfrak{u}_2^-$ is not locally transitive as well.

Step 4. Let $i=6$. The type of (L_6,L_6) is $\mathsf{A}_1+\mathsf{D}_5$, so $\dim L_6=49$, $\dim\mathfrak{u}_6^-=42$. By inspection of Φ^+ in [Bou, Planche VI] we obtain that, for the action of L_6 on $\mathfrak{u}_6^-$, there are exactly two shapes α_6 and $2\alpha_6$, and they determine the highest weights $-\alpha_6=(0,0,0,0,1,-2,1)$ and $-\alpha_2-\alpha_3-2\alpha_4-2\alpha_5-2\alpha_6-\alpha_7=(1,0,0,0,0,-1,0)$. Hence $\mathfrak{u}_6^-$ is the direct sum of two irreducible L_2-modules $\mathfrak{u}_{61}^-$ and $\mathfrak{u}_{62}^-$ that, considered as (irreducible) (L_2,L_2)-modules, have respectively the highest weights with the Dynkin diagrams



and


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If the action of L_6 on $\mathfrak{u}_6^-$ would be locally transitive, then all the more the action of $\mathbf{GL}_1\times L_6$ on $\mathfrak{u}_6^-$, where the first factor acts on $\mathfrak{u}_{61}^-$ by scalar multiplication and trivially on $\mathfrak{u}_{62}^-$, would be locally transitive. Using the notation and terminology of [KKIY], this in turn would mean that the pair $(\mathbf{GL}_1^2\times\mathbf{Spin}_{10}\times\mathbf{SL}_2, \Lambda'\otimes\Lambda_1+\Lambda_1\otimes 1)$ is 2-simple prehomogeneous of type I. However the classification of such pairs obtained in [KKIY, Section 3] shows that it is not so. Thus the action of L_6 on $\mathfrak{u}_6^-$ is not locally transitive.

Step 5. Let $i=7$. By Remark 4 and Corollary 2 of Proposition 2, the action of L_7 on $\mathfrak{u}_7^-$ is locally transitive. The type of (L_7,L_7) is E_6, so $\dim L_7=79$, $\dim\mathfrak{u}_7^-=27$. By the dimension reason, $\mathfrak{u}_7^-$ is a minimal irreducible (L_7,L_7)-module. Hence [SK, § 7, I, (27)] yields that $(L_7)_z^0$ for a general point $z\in\mathfrak{u}_7^-$ is a simple algebraic group of type F_4. Since the dimension of every nontrivial module of a simple group of type F_4 is at least 26, the $(L_7)_z^0$-module $\mathfrak{u}_7^-$ contains a trivial one-dimensional submodule and hence is not locally transitive. Lemma 4 now yields $\operatorname{ltd}(L_7:\mathfrak{u}_7^-)=1$. $\square$

11. $\mathbf{gtd}(L_i:\mathfrak{u}_i^-)$ for G of type E_8

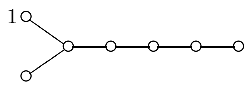
Proposition 12. *Let G be a connected simple algebraic group of type E_8. Then*

$$\operatorname{gtd}(L_i:\mathfrak{u}_i^-)=0 \quad \textit{for every} \quad i.$$

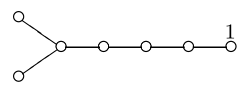
Proof. Step 1. We have $\dim G=248$. Let $i=2,3,4,5,6,7$. Then the type of (L_i,L_i) is respectively A_7, $A_1+\mathsf{A}_6$, $\mathsf{A}_1+\mathsf{A}_2+\mathsf{A}_4$, $\mathsf{A}_3+\mathsf{A}_4$, $\mathsf{A}_2+\mathsf{D}_5$, $\mathsf{A}_1+\mathsf{E}_6$. Hence respectively $\dim L_i=64,52,36,40,54,82$ and $\dim\mathfrak{u}_i^-=92,98,106,104,97,83$. As $\dim L_i<\dim\mathfrak{u}_i^-$, the action of L_i on $\mathfrak{u}_i^-$ is not locally transitive.

Step 2. Let $i=1$. The type of (L_1,L_1) is D_7, so $\dim L_1=92$, $\dim\mathfrak{u}_1^-=78$. By inspection of Φ^+ in [Bou, Planche VII] we obtain that, for the action of L_1 on $\mathfrak{u}_1^-$, there are exactly two shapes α_6 and $2\alpha_6$, and they determine the highest weights $-\alpha_1=(-2,0,1,0,0,0,0,0)$ and $-\alpha_1-2\alpha_2-3\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-\alpha_7=(-1,0,0,0,0,0,0,1)$. Hence $\mathfrak{u}_1^-$ is the direct sum of

two irreducible L_1-modules $\mathfrak{u}_{11}^-$ and $\mathfrak{u}_{12}^-$ that, considered as (irreducible) (L_1, L_1)-modules, have respectively the highest weights with the Dynkin diagrams



and



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If the action of L_1 on $\mathfrak{u}_1^-$ would be locally transitive, then all the more the action of $\mathbf{G}_m \times L_1$ on $\mathfrak{u}_1^-$, where the first factor acts on $\mathfrak{u}_{11}^-$ by scalar multiplication and trivially on $\mathfrak{u}_{12}^-$, would be locally transitive. This in turn would mean that, in the notation and terminology of [Kimu], $(\mathbf{GL}_1^2 \times \mathbf{Spin}_{14}, \Lambda' + \Lambda_1)$ is a prehomogeneous vector space with scalar multiplications. However the classification of such spaces obtained in [Kimu, Section 3] shows that it is not so. Thus the action of L_6 on $\mathfrak{u}_6^-$ is not locally transitive.

Step 3. Let $i = 8$. The type of (L_8, L_8) is E_7, so $\dim L_8 = 134$, $\dim u_8^- = 57$. By inspection of Φ^+ in [Bou, Planche VII] we obtain that, for the action of L_8 on $\mathfrak{u}_8^-$, there are exactly two shapes α_8 and $2\alpha_8$, and they determine the highest weights $-\alpha_8 = (0,0,0,0,0,0,1,-2)$ and $-2\alpha_8 = (0,0,0,0,0,0,0,-1)$. Hence $\mathfrak{u}_8^-$ is the direct sum of two irreducible L_8-modules $\mathfrak{u}_{81}^-$ and $\mathfrak{u}_{82}^-$ that, considered as (irreducible) (L_8, L_8)-modules, are respectively the unique 56-dimensional and a trivial 1-dimensional module. As the action of (L_8, L_8) on $\mathfrak{u}_{81}^-$ is not locally transitive, [SK, § 7], the same argument as in Subsection 10 for L_1 shows that the action of L_8 on $\mathfrak{u}_8^-$ is not locally transitive. $\square$

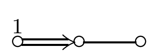
12. $\mathbf{gtd}(L_i : \mathfrak{u}_i^-)$ for G of type F_4

Proposition 13. *Let G be a connected simple algebraic group of type F_4. Then*

$$\operatorname{gtd}(L_i : \mathfrak{u}_i^-) = 0 \quad \textit{for every} \quad i.$$

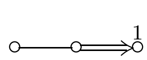
Proof. Step 1. We have $\dim G = 52$. For $i = 2, 3$ the type of (L_i, L_i) is $\mathsf{A}_1 + \mathsf{A}_2$, so $\dim L_i = 12$, $\dim \mathfrak{u}_i^- = 20$. As $\dim L_i < \dim \mathfrak{u}_i^-$, the action of L_i on $\mathfrak{u}_i^-$ is not locally transitive.

Step 2. Let $i = 1$. The type of (L_1, L_1) is C_3, so $\dim L_1 = 22$ and $\dim \mathfrak{u}_1^- = 15$. By inspection of Φ^+ in [Bou, Planche VIII] we obtain that, for the action of L_1 on $\mathfrak{u}_1^-$, there are exactly two shapes α_1 and $2\alpha_1$, and they determine the highest weights $-\alpha_1 = (-2,1,0,0)$ and $-2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4 = (-1,0,0,0)$. Hence $\mathfrak{u}_1^-$ is the direct sum of two irreducible L_1-modules $\mathfrak{u}_{11}^-$ and $\mathfrak{u}_{12}^-$, where $\mathfrak{u}_{11}^-$, considered as (irreducible) (L_1, L_1)-module, has the highest weight with the Dynkin diagram



and $\mathfrak{u}_{12}^-$ is a trivial 1-dimensional module. As the action of (L_1, L_1) on $\mathfrak{u}_{11}^-$ is not locally transitive, [SK, § 7], the same argument as in Subsection 10 for L_1 shows that the action of L_1 on $\mathfrak{u}_1^-$ is not locally transitive.

Step 3. Let $i = 4$. The type of (L_4, L_4) is B_3, so $\dim L_1 = 22$ and $\dim \mathfrak{u}_1^- = 15$. By inspection of Φ^+ in [Bou, Planche VIII] we obtain that, for the action of L_4 on $\mathfrak{u}_4^-$, there are exactly two shapes α_4 and $2\alpha_4$, and they determine the highest weights $-\alpha_4 = (0,0,1,-2)$ and $-\alpha_2 - 2\alpha_3 - 2\alpha_4 = (1,0,0,-2)$. Hence $\mathfrak{u}_4^-$ is the direct sum of two irreducible L_4-modules $\mathfrak{u}_{41}^-$ and $\mathfrak{u}_{42}^-$ that, considered as (irreducible) (L_4, L_4)-modules, have respectively the highest weights with the Dynkin diagrams



and

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From [SK, § 7, I, (16)] we now deduce that $(L_4)^0_z$ for a general point $z \in \mathfrak{u}^-_{41}$ is a simple algebraic group of type G_2. Hence $(L_4)^0_z \subset (L_4, L_4)$. As the action of (L_4, L_4) on $\mathfrak{u}^-_{42}$ is clearly not locally transitive, [SK, § 7], this yields that the action of $(L_4)^0_z$ on $\mathfrak{u}^-_{42}$ is not locally transitive as well. From Lemma 4 we then deduce that the action of L_4 on $\mathfrak{u}^-_4$ is not locally transitive. □

13. $\mathbf{gtd}(L_i : \mathfrak{u}_i^-)$ for G of type G_2

Proposition 14. *Let G be a connected simple algebraic group of type G_2. Then*

$$\operatorname{gtd}(L_i : \mathfrak{u}_i^-) = 0 \quad \textit{for every} \quad i.$$

Proof. We have $\dim G = 14$. For every i the type of (L_i, L_i) is A_1, so $\dim L_i = 4$, $\dim \mathfrak{u}_i^- = 5$. As $\dim L_i < \dim \mathfrak{u}_i^-$, the action of L_i on $\mathfrak{u}_i^-$ is not locally transitive. □

14. Proofs of Theorems 1–6

Proof of Theorem 5. Statement (i) follows from Proposition 2(iii), and (ii) from Propositions 6–14. □

Proof of Theorem 2. The claim follows from Proposition 3(ii). □

Proof of Theorem 4. Statement (a) follows from Proposition 2, and (b) from Theorem 5. □

Proof of Theorem 6. The argument is based on the following facts proved in [PV$_1$]. Let $\mathcal{O}(\varpi)$ be the G-orbit of a nonzero B-semi-invariant vector in $E(\varpi)$, and let $\mathcal{C}(\varpi)$ be the closure of $\mathcal{O}(\varpi)$ in $E(\varpi)$. Then $\mathcal{C}(\varpi)$ is a cone, i.e., stable with respect to the action of $\mathbf{G}_m$ on $E(\varpi)$ by scalar multiplications, and $\mathcal{O}(\varpi) = \mathcal{C}(\varpi) \setminus \{0\}$, [PV$_1$, Theorem 1]. This $\mathbf{G}_m$-action commutes with the G-action and yields a G-stable $\mathbb{Z}_+$-grading of the algebra $k[\mathcal{C}(\varpi)]$,

$$k[\mathcal{C}(\varpi)] = \textstyle\bigoplus_{n\in\mathbb{Z}_+} k[\mathcal{C}(\varpi)]_n. \tag{23}$$

For every $n \in \mathbb{Z}_+$, the G-module $k[\mathcal{C}(\varpi)]_n$ is isomorphic to $E(n\varpi^*)$, [PV$_1$, Theorem 2]. If ϖ is dominant, then $k[\mathcal{C}(\varpi)]$ is a unique factorization domain, [PV$_1$, Theorem 4].

Thus $G \times \mathbf{G}_m^d$ acts on $\mathcal{C}(\varpi)^d$, and $\mathcal{O}(\varpi)^d$ is an open $G \times \mathbf{G}_m^d$-stable subset of $\mathcal{C}(\varpi)^d$. Restricting the action to $\mathbf{G}_m^d$ yields a G-stable $\mathbb{Z}_+^d$-grading of the algebra $k[\mathcal{C}(\varpi)^d]$,

$$k[\mathcal{C}(\varpi)^d] = \textstyle\bigoplus_{(n_1,\ldots,n_d)\in\mathbb{Z}_+^d} k[\mathcal{C}(\varpi)^d]_{(n_1,\ldots,n_d)}. \tag{24}$$

Since $k[\mathcal{C}(\varpi)^d]$ and $k[\mathcal{C}(\varpi)]^{\otimes d}$ are isomorphic, (23) and (24) yield that the G-modules $k[\mathcal{C}(\varpi)^d]_{(n_1,\ldots,n_d)}$ and $E(n_1\varpi^*)\otimes\ldots\otimes E(n_d\varpi^*)$ are isomorphic for every $(n_1,\ldots,n_d) \in \mathbb{Z}_+^d$.

Let now $\pi_\varpi : \mathcal{O}(\varpi) \longrightarrow G/P(\varpi)$ be the natural projection. The G-equivariant morphism

$$\pi_\varpi^d :\ \mathcal{O}(\varpi)^d \longrightarrow \big(G/P(\varpi)\big)^d$$

is the quotient by $\mathbf{G}_m^d$-action. Hence it yields an isomorphism of invariant fields

$$k\big(\big(G/P(\varpi)\big)^d\big)^G \xrightarrow{\simeq} k\big(\mathcal{O}(\varpi)^d\big)^{G\times\mathbf{G}_m^d}. \tag{25}$$

Note that Definition 2 and Rosenlicht's theorem, [Ro], yield the equivalence

$$k\big(\big(G/P(\varpi)\big)^d\big)^G = k \iff \operatorname{gtd}(G : G/P(\varpi)) \geqslant d. \tag{26}$$

From (25) and (26) we deduce the equivalence

$$k\big(\mathcal{O}(\varpi)^d\big)^{G\times\mathbf{G}_m^d}=k \iff \operatorname{gtd}(G:G/P(\varpi))\geqslant d. \tag{27}$$

As $\mathcal{O}(\varpi)^d$ is open in $\mathcal{C}(\varpi)^d$, we have $k(\mathcal{O}(\varpi)^d)=k(\mathcal{C}(\varpi)^d)$. Hence (27) yields

$$k\big(\mathcal{C}(\varpi)^d\big)^{G\times\mathbf{G}_m^d}=k \iff \operatorname{gtd}(G:G/P(\varpi))\geqslant d. \tag{28}$$

We can now prove statements (i) and (ii) of Theorem 6.

(i) Assume the contrary. Take $n_1,\dots,n_d\in\mathbb{Z}_+$ such that

$$\dim\big(E(n_1\varpi^*)\otimes\ldots\otimes E(n_d\varpi^*)\big)^G\geqslant 2. \tag{29}$$

Then $\dim k[\mathcal{C}(\varpi)^d]^G_{(n_1,\dots,n_d)}\geqslant 2$, so there are nonzero elements $f_1,f_2\in k[\mathcal{C}(\varpi)^d]^G_{(n_1,\dots,n_d)}$ such that $f_1/f_2\notin k$. Since f_1 and f_2 are $\mathbf{G}_m^d$-semi-invariants of the same weight $(n_1,\dots n_d)$, we have $f_1/f_2\in k\big(\mathcal{C}(\varpi)^d\big)^{G\times\mathbf{G}_m^d}$. Thus $k\big(\mathcal{C}(\varpi)^d\big)^{G\times\mathbf{G}_m^d}\neq k$. By (28), this contradicts the condition $\operatorname{gtd}(G:G/P(\varpi))\geqslant d$.

(ii) Assume the contrary. Then by (28), there is a nonconstant rational function $f\in k\big(\mathcal{C}(\varpi)^d\big)^{G\times\mathbf{G}_m^d}$. Take now into account that (a) $k\big(\mathcal{C}(\varpi)^d\big)$ is the field of quotients of $k[\mathcal{C}(\varpi)^d]$ (as $\mathcal{C}(\varpi)^d$ is affine); (b) $k[\mathcal{C}(\varpi)^d]$ is a unique factorization domain (as ϖ is dominant); (c) $G\times\mathbf{G}_m^d$ is connected. By [PV$_2$, Theorem 3.3] these properties yield that $f=f_1/f_2$ for some $f_1,f_2\in k[\mathcal{C}(\varpi)^d]$ which are $G\times\mathbf{G}_m^d$-semi-invariants of the same weight. Since G has no nontrivial characters, the latter means that $f_1,f_2\in k[\mathcal{C}(\varpi)^d]^G_{(n_1,\dots,n_d)}$ for some $n_1,\dots,n_d\in\mathbb{Z}_+$. As $f_1/f_2\notin k$, this yields $\dim k[\mathcal{C}(\varpi)^d]^G_{(n_1,\dots,n_d)}\geqslant 2$. Hence (29) holds, and this contradicts (5). $\square$

Proof of Theorem 3. *Step* 1. If P_i is conjugate to P_i^- (i.e., $\varepsilon(\alpha_i)=\alpha_i$, see Section 4), then the claim follows from Proposition 2, its Corollary 1, and Theorem 5. This covers all but the following cases:

(a) G is of type A_l, and $2i\neq l+1$;
(b) G is of type D_l, l is odd, and $i=l-1,l$;
(c) G is of type E_6, and $i=1,3,5,6$.

Step 2. Consider case (b). By (11), we have $\operatorname{gtd}(G:G/P_{l-1})=\operatorname{gtd}(G:G/P_l)$. By Proposition 2, its Corollary 1, and Theorem 5, we have $\operatorname{gtd}(G:G/P_{l-1})\geqslant 3$.

Thus it suffices to prove $\operatorname{gtd}(G:G/P_{l-1})<4$. Towards this end we apply Theorem 6. First, note that for any semisimple G and $\lambda,\mu\in\mathrm{P}_{++}$, we have $E(\lambda)\otimes E(\mu)=\operatorname{Hom}(E(\lambda)^*,E(\mu))=\operatorname{Hom}(E(\lambda^*),E(\mu))$, and the elements of $\operatorname{Hom}(E(\lambda^*),E(\mu))^G$ are precisely G-module homomorphisms $E(\lambda^*)\to E(\mu)$. Since $E(\lambda^*)$ and $E(\mu)$ are simple, this yields

$$\dim\big(E(\lambda)\otimes E(\mu)\big)^G=\begin{cases}1 & \text{if } \lambda^*=\mu,\\ 0 & \text{otherwise.}\end{cases} \tag{30}$$

In case (b), we have $\varpi_s^*=\varpi_s$ for every $s\leqslant l-2$, whence by (30)

$$\dim\big(E(\varpi_s)\otimes E(\varpi_s)\big)^G=1 \quad \text{for every } s\leqslant l-2. \tag{31}$$

On the other hand, by [OV, Table 5], we have

$$E(\varpi_l)\otimes E(\varpi_l)=E(2\varpi_l)\oplus\textstyle\bigoplus_{i=1}^{\infty}E(\varpi_{l-2i}) \tag{32}$$

where, by definition, $\varpi_t = 0$ for $t < 0$. Since $l \geqslant 4$, from (31) and (32) it then clearly follows that

$$\dim\big(E(\varpi_l) \otimes E(\varpi_l) \otimes E(\varpi_l) \otimes E(\varpi_l)\big)^G \geqslant 2. \tag{33}$$

Since $E(\varpi_{l-1})^* = E(\varpi_l)$, Theorem 6(a) and (33) now yield $\operatorname{gtd}(G : G/P_{l-1}) < 4$. This completes the proof in case (b).

Step 3. Consider case (c). By (11), we have $\operatorname{gtd}(G : G/P_1) = \operatorname{gtd}(G : G/P_6)$ and $\operatorname{gtd}(G : G/P_3) = \operatorname{gtd}(G : G/P_5)$. Since $\dim G = 78$, $\dim G/P_6 = 16$, we have $\dim(G/P_6)^5 > \dim G$. Hence $\operatorname{gtd}(G : G/P_6) \leqslant 4$. By the Corollary of Lemma 5, we have $\operatorname{gtd}(G : G/P_5) \geqslant 2$. Thus it suffices to prove that $\operatorname{gtd}(G : G/P_6) \geqslant 4$ and $\operatorname{gtd}(G : G/P_5) < 3$. Towards this end we apply Theorem 6.

Namely, we have

$$\varpi_1^* = \varpi_6, \quad \varpi_3^* = \varpi_5. \tag{34}$$

By Theorem 6 and (34), proving $\operatorname{gtd}(G : G/P_6) \geqslant 4$ is equivalent to proving

$$\dim\big(E(n_1\varpi_1) \otimes E(n_2\varpi_1) \otimes E(n_3\varpi_1) \otimes E(n_4\varpi_1)\big)^G \leqslant 1 \quad \text{for every } n_1, \ldots, n_4 \in \mathbb{Z}_+. \tag{35}$$

To prove (35), we use that for every $r, s \in \mathbb{Z}_+$ the following decomposition holds (see [Li, 1.3]):

$$E(r\varpi_1) \otimes E(s\varpi_1) = \bigoplus_{\begin{cases} a_1, \ldots, a_4 \in \mathbb{Z}_+, \\ a_1 + a_3 + a_4 = r, \\ a_2 + a_3 + a_4 = s \end{cases}} E\big((a_1 + a_2)\varpi_1 + a_3\varpi_3 + a_4\varpi_6\big). \tag{36}$$

Since, by (34), we have $\big((a_1 + a_2)\varpi_1 + a_3\varpi_3 + a_4\varpi_6\big)^* = a_4\varpi_1 + a_3\varpi_5 + (a_1 + a_2)\varpi_6$, it follows from (36) and (30) that $\dim\big(E(n_1\varpi_1) \otimes E(n_2\varpi_1) \otimes E(n_3\varpi_1) \otimes E(n_4\varpi_1)\big)^G$ is equal to the number of solutions in $\mathbb{Z}_+$ of the following system of eight linear equations in eight variables $a_1, \ldots, a_4, b_1, \ldots, b_4$:

$$\begin{cases} a_4 = b_1 + b_2, \\ a_3 = 0, \\ b_3 = 0, \\ a_1 + a_2 = b_4, \\ a_1 + a_3 + a_4 = n_1, \\ a_2 + a_3 + a_4 = n_2, \\ b_1 + b_3 + b_4 = n_3, \\ b_2 + b_3 + b_4 = n_4. \end{cases}$$

Since this system is nondegenerate, there is at most one such solution. Thus, (35) holds; whence $\operatorname{gtd}(G : G/P_6) = 4$.

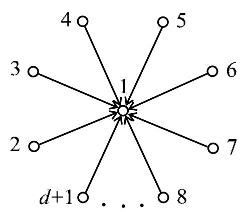
By Theorem 6 and (34), proving $\operatorname{gtd}(G : G/P_5) < 3$ is equivalent to proving

$$\dim\big(E(n_1\varpi_3) \otimes E(n_2\varpi_3) \otimes E(n_3\varpi_3)\big)^G \geqslant 2 \quad \text{for some } n_1, n_2, n_3 \in \mathbb{Z}_+. \tag{37}$$

Using Klimyk's formula, one checks that the decomposition of $E(2\varpi_3) \otimes E(2\varpi_3)$ into simple factors contains $E(2\varpi_5)$ with multiplicity 2 (using computer algebra system LiE, one obtains this decomposition in less than 1 second; this system is now available online at `http://wwwmathlabo.univ-poitiers.fr/~maavl/LiE/`). Hence, by (34) and (30), the inequality (37) holds for $n_1 = n_2 = n_3 = 2$. This completes the proof in case (c).

Step 4. To consider case (a), let now G be of type A_l and, more generally, no restrictions are imposed on i. By Proposition 4, we may (and shall) assume that (12) holds. Then G/P_i is the Grassmannian variety of i-dimensional linear subspaces of k^{l+1}.

Given an integer $d \geqslant 1$, let $\mathcal{V}_d$ be the quiver with $d+1$ vertices, d outside, one inside, and the arrows from each vertex outside to a vertex inside (the vertices are enumerated by $1, \dots, d+1$ so that the inside vertex is enumerated by 1):



Given a vector

$$\alpha := (a_1, \dots, a_{d+1}) \in \mathbb{Z}_+^{d+1},$$

we put $\mathbf{GL}_\alpha := \times_{i=1}^{d+1} \mathbf{GL}_{a_i}$ (we set $\mathbf{GL}_0 = \{e\}$). Let

$$\mathrm{Rep}(\mathcal{V}_d, \alpha) := \mathrm{Mat}_{a_1 \times a_2} \times \dots \times \mathrm{Mat}_{a_1 \times a_{d+1}}$$

be the space of α-dimensional representations of $\mathcal{V}_d$ endowed with the natural $\mathbf{GL}_\alpha$-action (we refer to [Ka$_1$], [Ka$_2$], [Ka$_3$], [DW] [Sch$_1$], [Sch$_2$] regarding the definitions and notions of the representation theory of quivers). For $\mathcal{V}_d$, the Euler inner product $\langle\ ,\ \rangle$ on $\mathbb{Z}^{d+1}$ is given by

$$\langle (x_1, \dots x_{d+1}), (y_1, \dots, y_{d+1}) \rangle = (x_1 y_1 + \dots + x_{d+1} y_{d+1}) - y_1 (x_2 + \dots + x_{d+1}). \tag{38}$$

It easily follows from the basic definitions, that the following properties are equivalent:

(i) there is an open G-orbit in $(G/P_i)^d$;
(ii) for $\gamma := (l+1, i, \dots, i)$, there is an open $\mathbf{GL}_\gamma$-orbit in $\mathrm{Rep}(\mathcal{V}_d, \gamma)$.

Note that by [Ka$_2$, Corollary 1 of Proposition 4], (ii) is equivalent to the following property

(iii) all the roots β_i appearing in the canonical decomposition of γ,
$$\gamma = \beta_1 + \dots + \beta_s, \tag{39}$$
are real, i.e., $\langle \beta_i, \beta_i \rangle = 1$.

Since there are combinatorial algorithms for finding decomposition (39), see [Sch$_1$], [Sch$_2$], [DW] (the algorithm in [DW] is very quick), (iii) and (38) permit, in principle, to check for every concrete d, l, i whether (i) holds or not, and thereby to calculate $\mathrm{gtd}(G : G/P_i)$. However we wish to obtain a closed formula for $\mathrm{gtd}(G : G/P_i)$. We preface the corresponding argument with the following observations.

(A) Let $\mathcal{V}_d^*$ be the quiver obtained from $\mathcal{V}_d$ by reversing the directions of all the arrows. Let

$$\mathrm{Rep}(\mathcal{V}_d^*, \alpha) := \mathrm{Mat}_{a_2 \times a_1} \times \dots \times \mathrm{Mat}_{a_{d+1} \times a_1}$$

be the space of α-dimensional representations of $\mathcal{V}_d^*$ endowed with the natural $\mathbf{GL}_\alpha$-action. The definition of $\mathbf{GL}_\alpha$-actions readily shows that restricting the map

$$\mathrm{Rep}(\mathcal{V}_d, \alpha) \longrightarrow \mathrm{Rep}(\mathcal{V}_d^*, \alpha), \quad (A_1, \dots, A_d) \mapsto (A_1^\top, \dots, A_d^\top),$$

to any $\mathbf{GL}_\alpha$-orbit in $\mathrm{Rep}(\mathcal{V}_d,\alpha)$ yields an isomorphism with some $\mathbf{GL}_\alpha$-orbit in $\mathrm{Rep}(\mathcal{V}_d^*,\alpha)$. Hence the existence of an open $\mathbf{GL}_\alpha$-orbit in $\mathrm{Rep}(\mathcal{V}_d,\alpha)$ is equivalent to its existence in $\mathrm{Rep}(\mathcal{V}_d^*,\alpha)$.

(B) If either $a_1\leqslant a_i$ for every i, or $a_1\geqslant a_2+\ldots+a_{d+1}$, then $\mathrm{Rep}(\mathcal{V}_d,\alpha)$ contains an open $\mathbf{GL}_\alpha$-orbit. This readily follows from the definition of $\mathbf{GL}_\alpha$-action on $\mathrm{Rep}(\mathcal{V}_d,\alpha)$.

(C) Let r_i be the ith fundamental reflection of $\mathbb{Z}^{d+1}$, i.e.,

$$r_i(\nu)=\nu-(\langle\nu,\alpha_i\rangle+\langle\alpha_i,\nu\rangle)\alpha_i,\quad \nu\in\mathbb{Z}^{d+1},\ \alpha_i=(0,\ldots,0,\underset{\hat{i}}{1},0\ldots,0). \tag{40}$$

It follows from (40), (38) that

$$r_i(\alpha)=\begin{cases}(-a_1+a_2+\ldots+a_{d+1},a_2,\ldots,a_{d+1}) & \text{for } i=1,\\ (a_1,\ldots,a_{i-1},a_1-a_i,a_{i+1},\ldots,a_{d+1}) & \text{for } i>1.\end{cases} \tag{41}$$

From [Ka$_1$, 2.3], [Ka$_3$], [SK, §2, Proposition 7], and (41) we then deduce the following.

(C$_1$) Let $a_1\leqslant a_2+\ldots+a_{d+1}$. Then $\mathrm{Rep}(\mathcal{V}_d,\alpha)$ contains an open $\mathbf{GL}_\alpha$-orbit if and only if $\mathrm{Rep}(\mathcal{V}_d^*,r_1(\alpha))$ contains an open $\mathbf{GL}_{r_1(\alpha)}$-orbit.

(C$_2$) Let $a_1\geqslant a_i$ for all $i>1$. Then $\mathrm{Rep}(\mathcal{V}_d,\alpha)$ contains an open $\mathbf{GL}_\alpha$-orbit if and only if $\mathrm{Rep}(\mathcal{V}_d^*,r_{d+1}\ldots r_2(\alpha))$ contains an open $\mathbf{GL}_{r_{d+1}\ldots r_2(\alpha)}$-orbit.

We can now complete the proof of Theorem 3 for G of type A_l using the argument due to A. Schofield, [Sch$_3$]. Namely, we shall show that for every $\lambda\in\mathfrak{G}:=\{(a_1,\ldots,a_{d+1})\in\mathbb{N}^{d+1}\mid a_2=\ldots=a_{d+1}\}$, we have

$$\mathrm{Rep}(\mathcal{V}_d,\lambda)\text{ contains an open }\mathbf{GL}_\lambda\text{-orbit}\iff\langle\lambda,\lambda\rangle>0. \tag{42}$$

By virtue of (2), Definition 2, and (38), this claim immediately yields the statement of Theorem 3 for G of type A_l since for $\lambda=(n,a,\ldots,a)$, we have

$$\langle\lambda,\lambda\rangle=n^2+da^2-nda. \tag{43}$$

Turning to the proof of claim, we call λ_1 and $\lambda_2\in\mathfrak{G}$ *congruent* if (i) $\langle\lambda_1,\lambda_1\rangle=\langle\lambda_2,\lambda_2\rangle$; (ii) $\mathrm{Rep}(\mathcal{V}_d,\lambda_1)$ contains an open $\mathbf{GL}_{\lambda_1}$-orbit $\iff$ $\mathrm{Rep}(\mathcal{V}_d,\lambda_2)$ contains an open $\mathbf{GL}_{\lambda_2}$-orbit. So, proving (42) is equivalent to that with λ replaced by a congruent vector. Recall that the quadratic form $\alpha\mapsto\langle\alpha,\alpha\rangle$ on $\mathbb{Z}^{d+1}$ is invariant with respect to the group generated by $r_1,\ldots,r_{d+1}$.

Take $\lambda=(n,a,\ldots,a)\in\mathfrak{G}$. If $a\geqslant n$, then by (43), $\langle\lambda,\lambda\rangle=n^2+da(a-n)>0$, and by (B), $\mathrm{Rep}(\mathcal{V}_d,\lambda)$ contains an open $\mathbf{GL}_\lambda$-orbit. This agrees with (42).

Assume now that $a<n$. Then by (C$_2$), (41), (A), vectors λ and $(n,n-a,\ldots,n-a)$ are congruent. Hence, in order to prove (42) we may (and shall) assume that

$$n\geqslant 2a. \tag{44}$$

Consider now separately two cases: $2n\leqslant da$ and $2n>da$. If

$$2n\leqslant da, \tag{45}$$

then from (38), (44), (45) we deduce

$$\begin{aligned}\langle\lambda,\alpha_1\rangle+\langle\alpha_1,\lambda\rangle&=2n-da\leqslant 0,\\ \langle\lambda,\alpha_i\rangle+\langle\alpha_i,\lambda\rangle&=2a-n\leqslant 0,\quad i>1.\end{aligned} \tag{46}$$

The inequalities (46) mean that λ lies in the fundamental set M of imaginary roots. By [Ka$_1$, Lemma 2.5], [Ka$_2$, Corollary 1 of Proposition 4], this yields that there is no open

$\mathbf{GL}_\lambda$-orbit in $\mathrm{Rep}(\mathcal{V}_d,\lambda)$. On the other hand, $\lambda\in M$ yields $\langle\lambda,\lambda\rangle\leqslant 0$. This agrees with (42).

Assume now that the second case holds, i.e., equivalently,

$$n > da - n. \tag{47}$$

If $n\geqslant da$, then, by (43), $\langle\lambda,\lambda\rangle = n(n-da)+da^2>0$, and, by (B), $\mathrm{Rep}(\mathcal{V}_d,\lambda)$ contains an open $\mathbf{GL}_\lambda$-orbit. This agrees with (42). Assume now that $da>n$. Then, by (C_1), (A), (47), vectors λ and $(da-n,a,\dots,a)$ are congruent. Thus, in view of (47), proving (42) is reduced to that with $\lambda=(n,a,\dots,a)$ replaced by $\lambda'=(n',a,\dots,a)$ where $0<n'<n$. We then can repeat the above arguments, starting from "Take $\lambda\ldots$", with λ replaced by λ'. Since the first coordinate of dimension vector strictly decreases via this process, the latter will eventually terminate. This completes the proof. $\square$

Proof of Theorem 1. Statements (i) and (ii) follow respectively from statements (ii) and (iii) of Proposition 1. Statement (iii) follows from Proposition 4, and (iv) from the Corollary of Proposition 5.

It is not difficult to deduce from (2) that $m_{li}\leqslant l+2$ for every $i\leqslant l$, and $m_{l1}=l+2$ (note that if $G=\mathbf{SL}_{l+1}$, then G/P_1 is $\mathbf{P}^l$ endowed with the natural $\mathbf{SL}_{l+1}$-action, and by Theorem 3, equality $m_{l1}=l+2$ expresses the well known elementary fact that this action is generically $(l+2)$-transitive). Statement (v) now follows from Definition 2, Theorem 2, and Theorem 3. $\square$

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